Math 120 HWK 5 Solutions: Six from 6.6

1. 6.6 p496 Problem 19. Find $\lim_{x\to\frac{\pi}{2}^-}(x-\frac{\pi}{2})\sec x$.

This limit is of the form $0\cdot\infty$, but it can easily be rewritten in the form $\frac{0}{0}$ since $\sec x$ is the reciprocal of $\cos x$. Doing so, and using l'Hospital's rule, we have

$$\lim_{x\to\frac{\pi}{2}^-}(x-\frac{\pi}{2})\sec x=\lim_{x\to\frac{\pi}{2}^-}\frac{x-\frac{\pi}{2}}{\cos x}=\lim_{x\to\frac{\pi}{2}^-}\frac{1}{-\sin x}=-1.$$

2. 6.6 p496 Problem 35. Find $\lim_{x\to 1^+}(\frac{1}{x-1}-\frac{1}{\ln x})$.

When x is slightly larger than 1, both $x-1$ and $\ln x$ are tiny positive numbers, so this limit is of the form $\infty-\infty$. Adding fractions allows us to rewrite it in the form $\frac{0}{0}$. Doing so, and using l'Hospital's rule three times, we have

$$\lim_{x\to 1^+}(\frac{1}{x-1}-\frac{1}{\ln x})=\lim_{x\to 1^+}\frac{\ln x-x+1}{x\ln x-\ln x}=\lim_{x\to 1^+}\frac{\frac{1}{x}-1}{1+\ln x-\frac{1}{x}}$$

$$=\lim_{x\to 1^+}\frac{1-x}{x+x\ln x-1}=\lim_{x\to 1^+}\frac{-1}{2+\ln x}=-\frac{1}{2}.$$

3. 6.6 p496 Problem 43. Find $\lim_{x\to 1^+}x^{\frac{1}{1-x}}$.

This is of the form $1^{-\infty}$, which is indeterminate. We'll use l'Hospital's rule to find $\lim_{x\to 1^+}\ln(x^{\frac{1}{1-x}})$ and then "e it up". We have

$$\lim_{x\to 1^+}\ln(x^{\frac{1}{1-x}})=\lim_{x\to 1^+}\frac{\ln x}{1-x}$$

The limit on the right is of the form $\frac{0}{0}$, so we use l'Hospital's rule to obtain

$$\lim_{x\to 1^+}\ln(x^{\frac{1}{1-x}})=\lim_{x\to 1^+}\frac{\ln x}{1-x}=\lim_{x\to 1^+}\frac{\frac{1}{x}}{-1}=-1.$$

Therefore $\lim_{x\to 1^+}x^{\frac{1}{1-x}}=\mathrm{e}^{-1}$ or $\frac{1}{\mathrm{e}}$.

4. 6.6 p496 Problem 55. Find $\lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan x}$, using some method other than l'Hospital's rule, which doesn't help.

The given limit, as written, is of the form $\frac{\infty}{\infty}$. What we can do here is to rewrite everything in terms of $\sin x$ and $\cos x$. Doing so and simplifying, we have

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan x} = \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\sin x} = 1$$

5. 6.6 p496 Problem 59. See text for statement of the problem.

Proposed solution (a) is faulty because $0 \cdot (-\infty)$ is undefined or indeterminate, not 0, and (b) is faulty for essentially the same reason: $0 \cdot (-\infty)$ is undefined or indeterminate, not $-\infty$. Proposal (c) is faulty for a similar reason: $\frac{-\infty}{\infty}$ is undefined, indeterminate, not -1. Fortunately, solution (d) is correct!

6. 6.6 pa496 Problem 61. Suppose $f(x)$ is given by $f(x) = \frac{9x - 3\sin 3x}{5x^3}$ for $x \neq 0$ and by $f(x) = c$ for $x = 0$. Find a value for c that makes the function f continuous at $x = 0$. Explain why your value of c works.

According to the definition of continuity, we only need to guarantee that $\lim_{x \to 0} f(x) = f(0)$. Since $f(0)$ is to equal the as yet unknown value of c, we need to have $c = \lim_{x \to 0} f(x)$. Using l'Hospital's rule (for the case $\frac{0}{0}$) three times in succession, we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{9x - 3\sin 3x}{5x^3} = \lim_{x \to 0} \frac{9 - 9\cos 3x}{15x^2} = \lim_{x \to 0} \frac{27\sin 3x}{30x} = \lim_{x \to 0} \frac{81\cos 3x}{30} = \frac{81}{30} = \frac{27}{10}$. So we need to have $c = \frac{27}{10}$.