Solutions to Homework Problems on Natural Fundamental Sets
Fall 2012, Math 246, Professor David Levermore

1. Given that e^{10t} and e^{-2t} is a fundamental set of solutions to $x'' - 8x' - 20x = 0$, find the solution $X(t)$ to the general initial-value problem

$$x'' - 8x' - 20x = 0\,, \qquad x(0) = x_0\,, \quad x'(0) = x_1\,.$$

Find the natural fundamental set of solutions associated with the initial time 0.

Solution. The solution of the general initial-value problem is

$$X(t) = \frac{2e^{10t} + 10e^{-2t}}{12}\,x_0 + \frac{e^{10t} - e^{-2t}}{12}\,x_1\,.$$

The natural fundamental set of solutions associated with the initial time 0 is thereby

$$N_0(t) = \frac{2e^{10t} + 10e^{-2t}}{12}\,, \qquad N_1(t) = \frac{e^{10t} - e^{-2t}}{12}\,.$$

2. Given that $e^{2t}\cos(t)$ and $e^{2t}\sin(t)$ is a fundamental set of solutions to $y'' - 4y' + 5y = 0$, find the solution $Y(t)$ to the general initial-value problem

$$y'' - 4y' + 5y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

Find the natural fundamental set of solutions associated with the initial time 0.

Solution. The solution of the general initial-value problem is

$$Y(t) = \big(e^{2t}\cos(t) - 2e^{2t}\sin(t)\big)\,y_0 + e^{2t}\sin(t)\,y_1\,.$$

The natural fundamental set of solutions associated with the initial time 0 is thereby

$$N_0(t) = e^{2t}\cos(t) - 2e^{2t}\sin(t)\,, \qquad N_1(t) = e^{2t}\sin(t)\,.$$

3. Given that t and t^3 is a fundamental set of solutions to $t^2z'' - 3tz' + 3z = 0$ for $t > 0$, find the solution $Z(t)$ to the general initial-value problem

$$t^2z'' - 3tz' + 3z = 0\,, \qquad z(2) = z_0\,, \quad z'(2) = z_1\,.$$

Find the natural fundamental set of solutions associated with the initial time 2.

Solution. The solution of the general initial-value problem is

$$Z(t) = \frac{12t - t^3}{16}\,z_0 + \frac{t^3 - 4t}{8}\,z_1\,.$$

The natural fundamental set of solutions associated with the initial time 2 is thereby

$$N_0(t) = \frac{12t - t^3}{16}\,, \qquad N_1(t) = \frac{t^3 - 4t}{8}\,.$$