ON VON KARMAN'S EQUATIONS AND THE BUCKLING OF A THIN ELASTIC PLATE

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Communicated by F. John, June 13, 1966

The object of this note is to demonstrate the applicability of the methods of nonlinear functional analysis in the investigation of a complex physical problem. In 1910 T. von Karman [9] introduced a system of 2 fourth order elliptic quasilinear partial differential equations which can be used to describe the large deflections and stresses produced in a thin elastic plate subjected to compressive forces along its edge. The most interesting phenomenon associated with this nonlinear situation is the appearance of "buckling," i.e. the plate may deflect out of its plane when these forces reach a certain magnitude. Mathematically this circumstance is expressed by the multiplicity of solutions of the boundary value problem associated with von Karman's equations. With the aid of the modern theory of linear elliptic partial differential equations together with functional analysis on a suitably chosen Hilbert function space, *we are able to use the structural pattern of the nonlinearity implicit in Karman's equations* to obtain a qualitative nonuniqueness theory for this problem.

Among the previous studies of buckling of plates are those of Friedrichs and Stoker [5] and Keller, Keller and Reiss [6], who study only radially symmetric solutions of circular plates. Numerical studies for rectangular plates have been given by Bauer and Reiss [2] among others. Karman's equations for general domains have been studied by Fife [4] and Morosov [8] in other connections. The authors are grateful to Professors S. Agmon and W. Littman for helpful suggestions. This research was partially supported by the National Science Foundation Grant No. GP-3904 and the Air Force Office of Scientific Research Grant No. 883-65.

1. **Classical and generalized solutions (for a clamped plate).** Let Ω be a bounded domain in R^2 with boundary $\partial\Omega$ consisting of a finite number of arcs on each of which a tangent rotates continuously. Defined over Ω, we consider the following system of partial differential equations and boundary conditions:

$$\Delta^2 f = -[w, w], \quad \Delta^2 w = \lambda[F, w] + [f, w], \tag{1}$$

$$w = w_x = w_y = f = f_x = f_y = 0 \quad \text{on} \quad \partial\Omega \tag{2}$$

where $[h, g] \equiv h_{xx}g_{yy} + h_{yy}g_{xx} - 2h_{xy}g_{xy}$ and Δ^2 denotes the biharmonic operator. This system is a version of von Karman's equations. Here w is a measure of the deflection of the plate and λ is a parameter measuring the magnitude of the compressive forces acting on $\partial\Omega$. The function F satisfies $\Delta^2 F = 0$, and λF represents the so-called stress function in an undeflected plate under the prescribed compressive forces. The actual stress function in the (possibly deflected) plate will be given by $\lambda F + f$. We shall assume throughout that $Lw = -[F, w]$ is uniformly elliptic, as is the case, for example, in the uniformly compressed plate where $Lw = \Delta w$.

DEFINITION 1. A classical solution of the system (1), (2) is a pair of functions (w, f) with the following properties:

(a) $w(x, y)$ and $f(x, y) \in C^4(\Omega) \cap C'(\bar{\Omega})$;

(b) $w(x, y)$ and $f(x, y)$ satisfy (1) and (2) pointwise.

We denote by $W_{2,2}(\Omega)$ the collection of all functions whose derivatives of all orders up to and including two lie in $L_2(\Omega)$. $W_{2,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_{2,2} = \sum_{|\alpha| \leq 2} (D^\alpha u, D^\alpha v)_{L_2(\Omega)}.$$

$\dot{W}_{2,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_{2,2}(\Omega)$.

DEFINITION 2. A generalized solution of the system (1), (2) is a pair of functions (u, g) with the following properties:

(a) $u(x, y)$ and $g(x, y) \in \dot{W}_{2,2}(\Omega)$

(b) u, g satisfy the following integral identities for all $\phi, \psi \in \dot{W}_{2,2}(\Omega)$:

$$a(g, \phi) = \lambda \int_\Omega (u_x u_{yy} \phi_x - u_x u_{xy} \phi_y); \tag{3a}$$

$$a(u, \psi) = \lambda \int_\Omega (\bar{g}_{xy} u_y - \bar{g}_{yy} u_x)\psi_x + (\bar{g}_{xy} u_x - \bar{g}_{xx} u_y)\psi_y; \tag{3b}$$

where $a(u, v) = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})$ and $\bar{g} = g + F$. The justification for the term "generalized solution" comes from the following result.

THEOREM 1. *Any classical solution of the system* (1), (2) *is a generalized solution* (*apart from constant mutiples*). *Conversely any generalized solution of* (1) *and* (2) *is a classical solution in* Ω *and at all sufficiently smooth portions of* $\partial\Omega$ (*apart from constant multiples*).

The former statement is a consequence of the fact that the boundary value problem (1) and (2) can be written in divergence form. A bootstrapping procedure using the L_p regularity theory of Agmon [1] can be used to prove the converse statement. The next result reduces the buckling problem to functional analysis.

LEMMA 1. *The generalized solutions of* (1), (2) *are identical with the solutions of an operator equation* $u-\lambda Cu=0$ *defined on a separable Hilbert space* H. *Here* C *is a compact nonlinear mapping of the form* L_0+T *where* L_0 *is a bounded compact self-adjoint linear operator mapping* H *into itself and* T *is a compact higher order operator with* $T(0)=0$.

This result is a consequence of the Sobolev Imbedding Theorem and the Riesz representation theorem for linear functionals on the Hilbert space $H=\dot{W}_{2,2}(\Omega)\times\dot{W}_{2,2}(\Omega)$.

2. **The linearized problem.** Associated with the system (1) and (2) we consider the following linear eigenvalue problem:

$$\begin{aligned}\Delta^2 w - \lambda[F, w] &= 0,\\ w = w_x = w_y &= 0 \quad \text{on} \quad \partial\Omega.\end{aligned} \tag{4}$$

LEMMA 2. *The spectrum of* (4) *consists of eigenvalues* $\{\lambda_n\}$ *forming a sequence of discrete positive numbers tending to* ∞ *with* n; *the multiplicity of each* λ_n *is finite.*

If Ω is a circle and $[F, w]=-\Delta w$ we note that each λ_n is simple, the first eigenfunction is radially symmetric, but not all eigenfunctions are radially symmetric. (If $\partial\Omega$ is not smooth, we interpret (4) in the generalized sense, as in §1).

3. **The multiplicity of solutions of the nonlinear problem.** The boundary value problem (1), (2) always has a solution, namely $w\equiv f\equiv 0$. We call this solution trivial and consider the multiplicity of solutions of (1) and (2) as a function of λ. If we denote by λ_n the eigenvalues of the linear problem (4), arranged in increasing order, then the following results are valid:

THEOREM 2. *For* $\lambda\leqq\lambda_1$, *there are no nontrivial generalized solutions of the system* (1), (2). *Furthermore nontrivial generalized solutions with sufficiently small norm can occur only for* λ *near one of the* λ_n.

THEOREM 3. *If* λ_n *is a simple eigenvalue of* (4), *there are positive numbers* ϵ_n *and* ρ_n *such that there are no nontrivial generalized solutions of the system* (1), (2) *with* $\|u\|_{2,2}+\|g\|_{2,2}\leqq\rho_n$, *in the closed interval* $[\lambda_n-\epsilon_n, \lambda_n]$; *and precisely two nontrivial generalized solutions* (u_λ, g_λ) *and* $(-u_\lambda, g_\lambda)$, *with* $\|u_\lambda\|_{2,2}+\|g_\lambda\|_{2,2}\leqq\rho_n$ *for each* λ *in the open interval* $(\lambda_n, \lambda_n+\epsilon_n)$.

THEOREM 4. *Suppose Ω is a circle of radius R, and $Lw=\Delta w$, (i.e. we consider uniform compression on $\partial\Omega$). If λ_n is an eigenvalue of* (3) *corresponding to a radially symmetric eigenfunction, then the generalized solutions of the nonlinear system referred to in Theorem* 3 *will also be radially symmetric. As a function of r these generalized solutions will have $(n-1)$ zeros on $[0, R]$.*

THEOREM 5. *Suppose λ_n is an eigenvalue of multiplicity m of* (3). *Then there are numbers ϵ_n, $\rho_n>0$ such that the system* (1), (2) *has no nontrivial generalized solutions with norm $\leqq\rho_n$ in the closed interval $[\lambda_n-\epsilon_n, \lambda_n]$. If m is odd, the system* (1), (2) *has at least two nontrivial generalized solutions with $\|u\|_{2,2}+\|g\|_{2,2}=\rho$ for each $\rho\leqq\rho_n$, in the open interval $(\lambda_n, \lambda_n+\epsilon_n)$.*

REMARKS ON PROOFS. Theorem 2 can be deduced from the variational characterization of λ_1 and the fact that $\int_\Omega[w, w]f=\int_\Omega[w, f]w$ (this identity was implicitly noted in Morosov [8]). Theorem 3 and the first part of Theorem 5 are obtained by modifying the operator equation of Lemma 1 to the form $w=\lambda(L_0w+Tw)$, $w\in\dot{W}_{2,2}(\Omega)$, where T is a nonlinear mapping, homogeneous of degree 3. A bifurcation procedure is then applied. The results of [6], together with the comments of §2 and Theorem 3, yield Theorem 4. The latter part of Theorem 5 is an immediate consequence of a computation of the topological degree of the mapping $I-\lambda C$ at $\lambda=\lambda_n-\epsilon_n$ and $\lambda=\lambda_n+\epsilon_n$ as in Krasnosel'skiĭ [7]. (Full proofs of these results will appear in a forthcoming paper.)

4. **Other boundary conditions.** Up to this point we have imposed Dirichlet boundary conditions (2) on w as well as f. This means the plate is "clamped" at its edge. We show now that other edge conditions can be treated in much the same way. But first we impose an extra assumption on the boundary $\partial\Omega$. We assume that no two of the boundary arcs are tangent at their intersection point (so that no corner is a cusp).

We now divide the boundary into three parts $\partial_1\Omega$, $\partial_2\Omega$, and $\partial_3\Omega$, each open relative to $\partial\Omega$, and such that $\partial\Omega=\bigcup\overline{\partial_i\Omega}$. We envision a plate which is "clamped" on $\partial_1\Omega$, "freely supported" on $\partial_2\Omega$, and "free" on $\partial_3\Omega$. The appropriate boundary conditions for classical solutions at a smooth edge would be:

$$w = w_x = w_y = 0 \qquad \text{on } \partial_1\Omega, \tag{5a}$$

$$w = B_1 w = 0 \qquad \text{on } \partial_2\Omega, \tag{5b}$$

$$B_1 w = B_2 w = 0 \qquad \text{on } \partial_3\Omega, \tag{5c}$$

and

$$B_1 w = w_{nn} + \sigma\left(w_{ss} - \frac{1}{\rho} w_n\right),$$

where

$$B_2 w = \Delta w_n + (1 - \sigma) w_{ssn},$$

σ is a constant, $-1<\sigma<1$, ρ is the radius of curvature, and subscripts n and s denote normal and tangential derivatives respectively. One final assumption on $\partial_i\Omega$ is needed:

Assumption on $\partial_i\Omega$: Either $\partial_1\Omega$ is nonvoid or $\partial_2\Omega$ does not lie on a straight line.

The above boundary conditions yield the definition of a classical solution. To define a generalized solution, let $\hat{W}_{2,2}(\Omega)$ be the closure in the norm of $\hat{W}_{2,2}$ of smooth functions vanishing on $\partial_2\Omega$ and in a neighborhood of $\partial_1\Omega$, and let

$$\hat{a}(u, v) = \int_\Omega \left[u_{xx}v_{xx} + (2 - 2\sigma)u_{xy}v_{xy} + u_{yy}v_{yy} + \sigma(u_{xx}v_{yy} + u_{yy}v_{xx})\right].$$

Bearing in mind the admissible range of σ, one may check from the above assumption on $\partial_i\Omega$ that $\hat{a}(u, v)$ is a legitimate scalar product for $\hat{W}_{2,2}$. A generalized solution is defined to be a pair $u\in\hat{W}_{2,2}(\Omega)$, $g\in\dot{W}_{2,2}(\Omega)$ satisfying (3) (with a replaced by $\hat{a}$ in the second equation only) for all $\phi\in\dot{W}_{2,2}(\Omega)$ and $\psi\in\hat{W}_{2,2}(\Omega)$. Again, a generalized solution can be shown to be classical up to smooth parts of the boundary interior to one of the $\partial_i\Omega$, and to satisfy the classical boundary conditions there. And again, the problem may be reduced to an equation $u-\lambda Cu=0$ in the Hilbert space $\dot{W}_{2,2}(\Omega)\times\hat{W}_{2,2}(\Omega)$.

The linearized problem is (4) with the new boundary conditions, and λ_1 is characterized by

$$\hat{a}(\phi, \phi) \geqq -\lambda_1(L\phi, \phi)$$

for all $\phi\in\hat{W}_{2,2}(\Omega)$ (the higher order boundary conditions arise as natural boundary conditions of the associated variational problem; see [**3**]).

THEOREM 6. *Under the above assumptions on* $\partial_i\Omega$, *all the preceding theorems and lemmas are true when Dirichlet conditions on w are replaced by* (5), *and the obvious changes are made in the statements of those theorems and lemmas.*

Added in proof. (September 21, 1966). Two extensions of the above results can be proven. First, in §1, we can replace the uniform ellipticity hypothesis on Lw by the assumption that Lw generates a compact linear operator in the appropriate Hilbert space H. This will be assured, for example, if all second derivatives of F are uniformly bounded in Ω. Secondly, Theorem 5 may be completed by eliminating any hypothesis on the parity of m. The proof of this fact requires a transformation of Von Karman equations into a variational formulation and an application of the Ljusternik-Schnirelmann theory of category. In particular if $m=2$ the system (1), (2) may have 6 nontrivial distinct solutions of small norm for each λ in the open interval $(\lambda_n, \lambda_n+\epsilon_n)$ if a certain numerical function of quantities related to the linearized problem (4) is positive.

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