CMPT 705 — Design and Analysis of Algorithms

Exercises on Randomized Algorithms. Due: Tuesday, November 22nd (at the beginning of the class)

Reminder: the work you submit must be your own. Any collaboration and consulting outside resources must be explicitly mentioned on your submission.

1. The result in the coupon collector problem does not imply that the probability of needing more than $cn \log n$ cereal boxes decreases sharply as c grows. Suppose that c is not a constant, but is allowed to grow with n. Let the probability of requiring more than $cn \log n$ trials be $p(c)$. For any c, show that $1/p(c)$ can be bounded from above and from below by polynomials in n.

2. Suppose you are presented with a very large set S of real numbers, and you would like to approximate the median of these numbers by sampling. You may assume all the members in S are distinct. Let $n = |S|$; we will say that a number x is an ε-approximate median of S if at least $(1/2-\varepsilon)n$ numbers in S are less than x, and at least $(1/2-\varepsilon)n$ numbers in S are greater than x. Consider an algorithm that works as follows. You select a subset $S' \subseteq S$ uniformly at random, compute the median of S', and return this as an approximate median of S. Show that there is an absolute constant c, independent of n, so that if you apply this algorithm with a sample S' of size c, then with probability at least .99, the number returned will be a (.05)-approximate median of S. (You may consider either the version of the algorithm that constructs S' by sampling with replacement, so that an element of S can be selected multiple times, or one without replacement.)

3. Consider the following analogue of Karger's algorithm for finding minimum $s-t$ cuts. We will contract edges iteratively using the following randomized procedure. In a given iteration, let s and t denote the possibly contracted nodes that contain the original nodes s and t, respectively. To make sure that s and t do not get contracted, at each iteration we delete any edges connecting s and t and select a random edge to contract among the remaining edges. Give an example to show that the probability that this method finds a minimum $s-t$ cut can be exponentially small.