On Highly Robust Efficient Solutions to Uncertain Multiobjective Linear Programs

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Abstract

Decision making in the presence of uncertainty and multiple conflicting objectives is a real-life issue in many areas of human activity. To address this type of problem, we study highly robust efficient solutions to uncertain multiobjective linear programs (UMOLPs) with objective-wise uncertainty in the objective function coefficients. We develop properties of the highly robust efficient set, provide its characterization using the cone of improving directions associated with the UMOLP, derive several bound sets on the highly robust efficient set, and present a robust counterpart for a class of UMOLPs. As various results rely on the acuteness of the cone of improving directions, we also propose methods to verify this property.

Keywords: multiple objective programming, robust multiobjective optimization, objective-wise uncertainty, polar cones, acute cones

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1. Introduction

Practical problems often involve conflicting goals and uncertainty present during the decision-making process. Problems with conflicting criteria typically do not have a unique optimal solution, and multiobjective optimization instead provides a solution set of alternatives that is indispensable in revealing a compromise. Independently of conflict, problems involving uncertainty require *robust solutions* that are desirable in some sense across all realizations of uncertainty. To account for uncertainty in single-objective optimization, Ben-Tal & Nemirovski (1998) developed a deterministic methodology known as robust optimization that uses crisp sets to define regions within which the

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uncertainty varies. Solutions are computed by solving a *robust counterpart* (RC), which is a related deterministic problem whose data is certain or determined.

In the field of robust multiobjective optimization, which integrates robust and multiobjective optimization, uncertainty may be associated with multiobjective programs (MOPs) in a variety of ways yielding an uncertain MOP (UMOP). Depending on the sources of uncertainty that are considered, the formulation changes to reflect the specific situation. For each formulation, a variety of robustness concepts are defined and studied. Refer to Wiecek & Dranichak (2016) for a tutorial. That being said, in this paper, we focus on uncertain multiobjective linear programs (UMOLPs) in which only objective coefficients are uncertain (and the feasible set is deterministic) as in, e.g., Ehrgott et al. (2014), Ide & Schöbel (2015), and Kuhn et al. (2016).

The following notation is used throughout. The Euclidean space of dimension n is given by $\mathbb{R}^n$. The closure of a set $V \subseteq \mathbb{R}^n$ is denoted by $\mathrm{cl}(V)$, the interior by $\mathrm{int}(V)$, the relative interior by $\mathrm{rel\,int}(V)$, and the dimension by $\dim(V)$. The Minkowski sum of two sets $V, W \subseteq \mathbb{R}^n$ is given by $V \oplus W := \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in V, \mathbf{w} \in W\}$, and the Cartesian product by $V \times W := \{(\mathbf{v}, \mathbf{w}) : \mathbf{v} \in V, \mathbf{w} \in W\}$. The rank of a matrix $\mathbf{M} \in \mathbb{R}^{p \times n}$ is denoted by $\mathrm{rank}(\mathbf{M})$, and the vector of all zeros by $\mathbf{0}$. For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we write $\mathbf{v} \leqq \mathbf{w}$ if $v_j \leq w_j$ for all $j = 1, \ldots, n$; $\mathbf{v} \leq \mathbf{w}$ if $v_j \leq w_j$ for all $j = 1, \ldots, n$, and $\mathbf{v} \neq \mathbf{w}$; and $\mathbf{v} < \mathbf{w}$ if $v_j < w_j$ for all $j = 1, \ldots, n$. When $n = 1$, the symbols $\leqq$ and $\leq$ coincide. The symbols $\geqq, \geq, >$ are used similarly.

A deterministic MOLP is a problem of the form:

$$\min_{\mathbf{x} \in X} \quad \mathbf{C}\mathbf{x} = \begin{bmatrix} \mathbf{c}_1\mathbf{x} & \cdots & \mathbf{c}_p\mathbf{x} \end{bmatrix}^T \tag{1.1}$$

where $\mathbf{c}_k, k = 1, \ldots, p$, is the k-th row of the $p \times n$ *cost (objective) matrix* $\mathbf{C}, p \geq 2, n \geq 1$, $\mathbf{x} \in \mathbb{R}^n$ is the *decision vector*, $X := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\} \subset \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$, is the (bounded polyhedral) *feasible region*, and $\mathbb{R}^n$ is the *decision (solution) space*.

The commonly used solution concept for MOLP (1.1) is that of Pareto efficiency, or simply efficiency. A feasible solution $\hat{\mathbf{x}} \in X$ to MOLP (1.1) is said to be *(weakly) efficient* if there does not exist $\mathbf{x} \in X$ such that $\mathbf{C}\mathbf{x}$ $(<)$ $\leq$ $\mathbf{C}\hat{\mathbf{x}}$. The set of all (weakly) efficient solutions $\hat{\mathbf{x}} \in X$ is denoted by ($\mathrm{wE}(X, \mathbf{C})$) $\mathrm{E}(X, \mathbf{C})$ and is called the *(weakly) efficient set*. Since we assume that X is bounded, (weakly) efficient solutions to MOLP (1.1) are guaranteed to exist (see Corollary 2.26 and Theorem 2.19, Ehrgott 2005, respectively).

Considering uncertain input data in the cost matrix coefficients of MOLP (1.1), we obtain a

UMOLP. We define a UMOLP, denoted MOLP(U), to be the collection of MOLPs, which are denoted by MOLP($\mathbf{u}$), indexed by the (uncertain) parameter $\mathbf{u}$:

$$\left\{ \min_{\mathbf{x}\in X} \quad \mathbf{C}(\mathbf{u})\mathbf{x} \right\}_{\mathbf{u}\in U,} \tag{1.2}$$

where $U \subseteq \mathbb{R}^q$ is a nonempty set modeling the uncertainty referred to as the *uncertainty set* or *set of scenarios*, and $\mathbf{C}(\mathbf{u})$ is the *cost matrix under uncertainty* $\mathbf{u} \in U$. Every problem MOLP($\mathbf{u}$) in the collection, which is called an *instance* of MOLP(U), is associated with a particular value of $\mathbf{u} \in U$ that is referred to as a *realization* or *scenario*. Note that if the set of scenarios U is a singleton, then the uncertain problem (1.2) reduces to the deterministic problem (1.1). While the solution concept for MOLP(U) is not obvious, the concept for each instance is clear since MOLP($\mathbf{u}$) is a deterministic MOLP given the scenario $\mathbf{u} \in U$. Accordingly, (wE($X, \mathbf{C}(\mathbf{u})$)) E($X, \mathbf{C}(\mathbf{u})$) denotes the (weakly) efficient set of MOLP($\mathbf{u}$) for some realization $\mathbf{u} \in U$.

In practical problems, conflicting objective functions are unlikely to depend on the same uncertainties. To accommodate this reality, we assume that the uncertainty, or MOLP(U), is *objective-wise*. In particular, MOLP(U) is said to be *objective-wise* if the uncertainties of the cost vectors $\mathbf{c}_1, \dots, \mathbf{c}_p$ are independent of each other, that is, if $U = U_1 \times \cdots \times U_p$, where $U_k \subseteq \mathbb{R}^n, k = 1, \dots, p$, such that

$$\mathbf{C}(\mathbf{u}) = \begin{bmatrix} \mathbf{c}_1(\mathbf{u}_1) \\ \vdots \\ \mathbf{c}_p(\mathbf{u}_p) \end{bmatrix} = \begin{bmatrix} c_{11}u_{11} & \cdots & c_{1n}u_{1n} \\ \vdots & \ddots & \vdots \\ c_{p1}u_{p1} & \cdots & c_{pn}u_{pn} \end{bmatrix} \tag{1.3}$$

with $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_p \end{bmatrix}^T \in U$, and $\mathbf{u}_k \in U_k, k = 1, \dots, p$. Based on the formulation of $\mathbf{C}(\mathbf{u})$, it is easy to show that the function $\mathbf{C}(\mathbf{u})\mathbf{x}$ is bilinear in $\mathbf{x}$ and $\mathbf{u}$.

Perhaps the first type of objective-wise UMOLP with only uncertain objective coefficients considered in the literature is from the field of interval multiobjective programming in which the uncertain coefficients fall within a closed interval that is assumed to be known. In Bitran (1980), an interval MOLP (IMOLP) is the collection of MOLPs indexed by the cost matrix $\mathbf{C}$:

$$\left\{ \min_{\mathbf{x}\in X} \quad \mathbf{C}\mathbf{x} \right\}_{\mathbf{C}\in \Phi,} \tag{1.4}$$

where $\Phi \subseteq \mathbb{R}^{p\times n}$ is the nonempty set of $p \times n$ matrices with elements $c_{kj} \in [c^{\mathrm{L}}_{kj}, c^{\mathrm{U}}_{kj}], k = 1, \dots, p, j = 1, \dots, n$. The lower bounds c^{L}_{kj} and upper bounds c^{U}_{kj} are assumed to be known.

It is clear that all IMOLPs can be reformulated as objective-wise UMOLPs by taking $c_{kj} = 1$ in (1.3) for all $k = 1, \ldots, p, j = 1, \ldots, n$, and $U_k = \{\mathbf{u}_k \in \mathbb{R}^n : c_{k1}^{\mathrm{L}} \leq u_{k1} \leq c_{k1}^{\mathrm{U}}, \ldots, c_{kn}^{\mathrm{L}} \leq u_{kn} \leq c_{kn}^{\mathrm{U}}\}, k = 1, \ldots, p$, which is often referred to as a *box uncertainty set*. On the other hand, it is equally clear that not all objective-wise UMOLPs can be reformulated as IMOLPs, which is the case, for instance, when U is finite. As an example, consider

$$\left\{ \min_{\mathbf{x} \in X} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \mathbf{x} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2}, \tag{1.5}$$

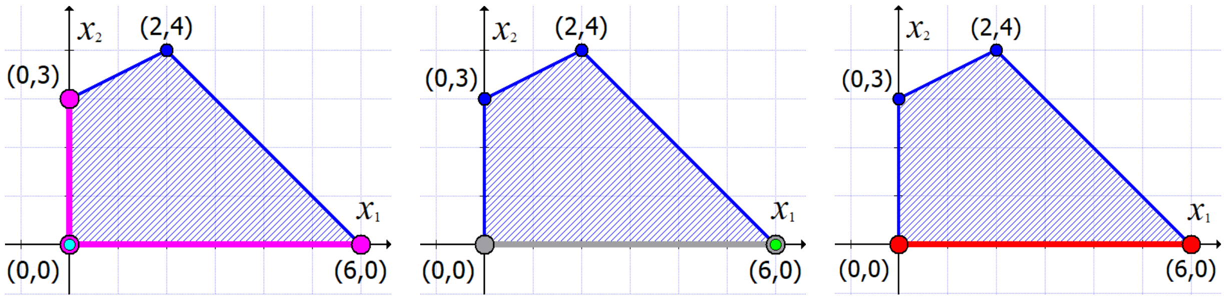
where $U_1 = \{(1, 1), (2, 3)\}$, and $U_2 = \{(1, 2)\}$. Here, we have that $U = \{(1, 1, 1, 2), (2, 3, 1, 2)\}$. As UMOLP (1.5) is a collection of two MOLPs, it cannot possibly be reformulated as an IMOLP that is necessarily an infinite collection of MOLPs or a singleton (if $c_{kj}^{\mathrm{L}} = c_{kj}^{\mathrm{U}}$ for all k and j). Since all IMOLPs can be reformulated as objective-wise UMOLPs with box uncertainty sets, which is only one of many possible types of uncertainty sets, and UMOLPs with finite uncertainty sets cannot be reformulated as IMOLPs, it is evident that UMOLP (1.2) is more general than IMOLP (1.4) and permits a wider variety of problems to study.

To solve objective-wise UMOLPs with uncertain objective function coefficients, a variety of possible solution concepts may be chosen. For a comprehensive survey of ten different concepts of robust efficiency for this type of problem and their numerous relationships, refer to Ide & Schöbel (2015). We choose to adopt the conservative concept of *necessary efficiency* (see Inuiguchi & Kume 1991) that is first mentioned in 1980 by Bitran (1980). Such solutions are efficient with respect to every realization of the uncertain data. However, in keeping with the recent literature on robust multiobjective optimization, we refer to these solutions as *highly robust efficient*.

Definition 1.1. A solution $\mathbf{x}^* \in X$ to MOLP(U) is said to be *highly robust (weakly) efficient* (HR(W)E) if for every $\mathbf{u} \in U$, there does not exist $\mathbf{x} \in X$ such that $\mathbf{C}(\mathbf{u})\mathbf{x}\ (<) \leq \mathbf{C}(\mathbf{u})\mathbf{x}^*$.

In other words, a solution $\mathbf{x}^* \in X$ is an HR(W)E solution to MOLP(U) if and only if ($\mathbf{x}^* \in \bigcap_{\mathbf{u} \in U} \mathrm{wE}(X, \mathbf{C}(\mathbf{u}))$) $\mathbf{x}^* \in \bigcap_{\mathbf{u} \in U} \mathrm{E}(X, \mathbf{C}(\mathbf{u}))$ (see p. 242, Ide & Schöbel 2015). The *highly robust (weakly) efficient set* of MOLP(U) is denoted by ($\mathrm{wE}(X, \mathbf{C}(\mathbf{u}), U)$) $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$.

Note that, as in the deterministic setting with efficiency, the HRE set is contained in the HRWE set. However, it is important to recognize a key difference between the solutions to deterministic and uncertain MOLPs. In the deterministic case, provided that X is bounded, the weakly efficient and



(a) The weakly efficient (purple) and efficient (teal) sets under $\mathbf{u}^1 = (1, 0, 0, 1)$

(b) The weakly efficient (grey) and efficient (green) sets under $\mathbf{u}^2 = (-1, 0, 0, 1)$

(c) The HRWE set (red)

Figure 1: Weakly efficient, efficient, and HRWE points of UMOLP (1.5) with $U_1 = \{(1, 0), (-1, 0)\}, U_2 = \{(0, 1)\}$, and feasible set X_1

efficient sets of MOLP (1.1) are nonempty (cf. Corollary 2.26 and Theorem 2.19, Ehrgott 2005). On the other hand, in the uncertain case, the HRWE set of UMOLP (1.2) may be nonempty while the HRE set is empty. For example, consider UMOLP (1.5) with $U_1 = \{(1, 0), (-1, 0)\}, U_2 = \{(0, 1)\}$, and the bounded feasible set (refer to Example 1, Wiecek & Dranichak 2016) given by

$$X_1 := \{\mathbf{x} \in \mathbb{R}^2 : -x_1 + 2x_2 \leq 6, x_1 + x_2 \leq 6, x_1 \geq 0, x_2 \geq 0\}. \tag{1.6}$$

We have, as shown in Figure 1c, that the HRWE set is nonempty while the HRE set is empty. With this in mind, we only address HRWE solutions in certain cases.

In the interval multiobjective programming literature, solution methods for computing necessarily efficient solutions to IMOLPs are presented by Bitran (1980), Benson (1985), Inuiguchi & Kume (1992), Ida (1996), Inuiguchi & Sakawa (1996), Oliveira & Antunes (2007), and Hladík (2010), while complexity analysis is studied by Hladík (2012). However, as IMOLPs are a special case of objective-wise UMOLPs, it is desirable to study the more general context.

Independently of the interval multiobjective programming studies, in recent years, Ide & Schöbel (2015), Goberna et al. (2015), Kuhn et al. (2016), and Wiecek & Dranichak (2016) study HRE solutions to UMOPs. In particular, Kuhn et al. (2016) examine the special case of an uncertain biobjective problem with only one uncertain objective. Meanwhile, when the uncertainty set is both objective-wise and a bounded polyhedron and the objective functions are affine with respect to the uncertainty, Ide & Schöbel (2015) prove that solving a UMOP with respect to the original uncertainty set is equivalent to solving the UMOP with respect to the finite set of extreme points of the uncertainty set (see Theorem 46, Ide & Schöbel 2015). Moreover, the authors show in Example 48 that this theorem does not hold if the assumption of objective-wise uncertainty is relaxed.

In view of this work, the use of objective-wise uncertainty takes on more significance and, since $\mathbf{C}(\mathbf{u})\mathbf{x}$ is linear in $\mathbf{u}$, we restrict our attention to finite uncertainty sets for the remainder of the paper. Although certain results may also be true for general infinite uncertainty sets, we do not address this in more detail. Throughout, the finite set of scenarios is defined to be

$$U := \{\mathbf{u}^1, \dots, \mathbf{u}^s\} \subset \mathbb{R}^q,$$

where we assume WLOG that each scenario is distinct.

The state of the art in research on highly robust efficiency offers several research directions that we undertake in this paper. Our first contribution is a characterization of the HRE set by means of the strict polar cones of the cones of improving directions associated with every instance of MOLP(U). This characterization is a consequence of the preliminaries that we present on polars and strict polars of convex cones and unions of convex cones. Here, the strict polar of a single convex cone is used in the context of MOLP (1.1) (or a single-scenario UMOLP) and efficiency, while the strict polar of a union of cones is used in the context of MOLP(U) (associated with a collection of scenarios) and highly robust efficiency. A second contribution is several bound sets on the HRE set whose associated MOLPs lead to an RC of MOLP(U) and a condition for the existence of HRE solutions.

The paper is organized as follows. In Section 2, we develop theory on polar and strict polar cones in preparation for the main results given in Section 3 regarding the characterization of and bound sets on the HRE set. Concluding remarks are given in Section 4.

2. Preliminary Results

Since each instance of MOLP(U) is a deterministic problem, we briefly review two relevant deterministic efficiency results, one of which we reformulate using the language of and theory regarding three related cones. Before we accomplish this in Section 2.2, we derive algebraic representations of the polars and strict polars of these cones in Section 2.1. In addition, since highly robust efficiency relies on efficiency to each instance, we consider finite collections of the same three cones in Section 2.3 and derive formulas for the polars and strict polars of unions of these collections. All of this theory becomes useful when developing a characterization of the HRE set in Section 3.

2.1. On Polars and Strict Polars of Convex Cones

A set $K \subseteq \mathbb{R}^n$ is called a *cone* if $\mathbf{x} \in K$ implies that $\lambda \mathbf{x} \in K$ for all $\lambda > 0$. Accordingly, cones do not have to contain the origin. A cone $K \subseteq \mathbb{R}^n$ is called *acute* if $\mathrm{cl}(K) \subseteq H \cup \{\mathbf{0}\}$, where H is an open half-space whose generating hyperplane passes through the origin; *pointed* if $\mathbf{x} \in K$ and $\mathbf{x} \neq \mathbf{0}$ implies that $-\mathbf{x} \notin K$; and *convex* if for any two points $\mathbf{x}_1, \mathbf{x}_2 \in K$, then $\mathbf{x}_1 + \mathbf{x}_2 \in K$. Relating acute and pointed cones, we have the following proposition.

Proposition 2.1. *Let $K \subseteq \mathbb{R}^n$ be a cone. If K is acute, then it is pointed.*

Proof. Let K be acute. By definition, there is an open half-space H generated by the hyperplane passing through the origin, $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = 0\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{a} \neq \mathbf{0}$, such that $\mathrm{cl}(K) \subseteq H \cup \{\mathbf{0}\}$. WLOG, we have $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} > 0\}$, and $\mathrm{cl}(K) \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} > 0\} \cup \{\mathbf{0}\}$.

Now, assume that K is not pointed. By definition, there exists an $\mathbf{x} \in K, \mathbf{x} \neq \mathbf{0}$, such that $-\mathbf{x} \in K$. Since $\mathbf{x} \in K \subseteq \mathrm{cl}(K)$ and $\mathbf{x} \neq \mathbf{0}$, we know that $\mathbf{a}^T\mathbf{x} > 0$. Similarly, since $-\mathbf{x} \in K \subseteq \mathrm{cl}(K)$ and $-\mathbf{x} \neq \mathbf{0}$, we know that $\mathbf{a}^T(-\mathbf{x}) > 0$, which gives $\mathbf{a}^T\mathbf{x} < 0$, a contradiction. $\square$

With regards to convex cones, one type we use in our study is the normal cone. The *normal cone* to X at $\bar{\mathbf{x}} \in X$ is a convex cone defined to be $N_X(\bar{\mathbf{x}}) := \{\mathbf{p} \in \mathbb{R}^n : \mathbf{p}^T(\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \text{ for all } \mathbf{x} \in X\}$. Note that the normal cone $N_X(\mathbf{x})$ contains $\mathbf{0}$ for all $\mathbf{x} \in X$ and is thus always nonempty.

In addition, we examine finite and polyhedral convex cones. A nonempty convex cone $K \subseteq \mathbb{R}^n$ is called *finite* if for $\mathbf{g}_j \in \mathbb{R}^n, j = 1, \ldots, r$, then $K := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{j=1}^r \lambda_j \mathbf{g}_j, \lambda_j \geq 0, j = 1, \ldots, r\}$. Here, K is said to be *spanned* or *generated* by the finite *set of generators* $\{\mathbf{g}_1, \ldots, \mathbf{g}_r\}$. Moreover, a nonempty convex cone $K \subseteq \mathbb{R}^n$ is called *polyhedral convex* if it is the intersection of a finite number of closed half-spaces whose generating hyperplanes pass through the origin. The well-known *Minkowski-Weyl Theorem* (see Theorem 4.7.2, Panik 1993) relates finite cones and polyhedral convex cones in that a nonempty cone $K \subseteq \mathbb{R}^n$ is polyhedral convex if and only if it is finite. In view of the Minkowski-Weyl Theorem, every polyhedral convex cone has two representations: (i) *generator form* $K(\mathbf{G}^T) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$, where $\mathbf{G}^T = \begin{bmatrix} \mathbf{g}_1 & \cdots & \mathbf{g}_r \end{bmatrix} \in \mathbb{R}^{n \times r}$ and $\{\mathbf{g}_1, \ldots, \mathbf{g}_r\}$ is a finite set of generators of K, and (ii) *inequality form* $K_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} \leqq \mathbf{0}\}$, where

$\mathbf{M} \in \mathbb{R}^{p\times n}$ and the rows of $\mathbf{M}$ are the normals to the generating hyperplanes whose half-spaces form K. We may convert between the two forms using, e.g., SageMath's `polyhedron base class`, The Sage Developers (2017). As is clear in the inequality form representation of a polyhedral convex cone, $K_{\leqq}(\mathbf{M})$ contains $\mathbf{0}$ for all $\mathbf{M}$ and is thus always nonempty.

Two convex cones closely related to polyhedral convex cones are $K_{\leq}(\mathbf{M}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} \leq \mathbf{0}\}$ and $K_{<}(\mathbf{M}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M}\mathbf{x} < \mathbf{0}\}$. It is clear that $K_{<}(\mathbf{M})$ is open, while $K_{\leq}(\mathbf{M})$ may be open, closed, or neither. Although $K_{\leqq}(\mathbf{M})$ is always nonempty, $K_{\leq}(\mathbf{M})$ and $K_{<}(\mathbf{M})$ may be empty.

For a cone $K \subseteq \mathbb{R}^n$, not necessarily convex, its *polar cone* is the set $K^+ := \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} \geq 0 \text{ for all } \mathbf{x} \in K\}$, and its *strict polar cone* is the set $K^{s+} := \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} > 0 \text{ for all } \mathbf{x} \in K \setminus \{\mathbf{0}\}\}$. Note that if $K \neq \emptyset$, then $K^+ \neq \emptyset$ as well (as it must contain $\mathbf{0}$), while K^{s+} may be empty.

We derive the polars and strict polars of the convex cones $K_{\leqq}(\mathbf{M}), K_{\leq}(\mathbf{M})$, and $K_{<}(\mathbf{M})$, which we need for the deterministic case of efficiency. Given the cones $K_{\leqq}(\mathbf{M})$, $K_{\leq}(\mathbf{M})$, and $K_{<}(\mathbf{M})$, we denote their polars by $K^+_{\leqq}(\mathbf{M})$, $K^+_{\leq}(\mathbf{M})$, and $K^+_{<}(\mathbf{M})$, and their strict polars by $K^{s+}_{\leqq}(\mathbf{M})$, $K^{s+}_{\leq}(\mathbf{M})$, and $K^{s+}_{<}(\mathbf{M})$, respectively. The algebraic representations of the polars of the three convex cones, which are equivalent under a specific condition, are given in the following.

Proposition 2.2. *(i) The equality* $K^+_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *holds.*

(ii) Let $\mathrm{cl}(K_{<}(\mathbf{M})) = \mathrm{cl}(K_{\leq}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$. *Then* $K^+_{<}(\mathbf{M}) = K^+_{\leq}(\mathbf{M}) = K^+_{\leqq}(\mathbf{M})$.

Proof. (i) Given by Proposition 2.1.13, Sawaragi et al. (1985).

(ii) Follows directly from Proposition 2.1.5(iii), Sawaragi et al. (1985). □

Since $K_{\leqq}(\mathbf{M}) \neq \emptyset$ and $\mathrm{cl}(K_{<}(\mathbf{M})) = \mathrm{cl}(K_{\leq}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$, it follows that $K_{<}(\mathbf{M}) \neq \emptyset$ and $K_{\leq}(\mathbf{M}) \neq \emptyset$ as well. Hence, the polars derived in Proposition 2.2 are always nonempty. Similarly, under certain assumptions, it is clear in the following derivation that $K^{s+}_{\leqq}(\mathbf{M}), K^{s+}_{\leq}(\mathbf{M})$, and $K^{s+}_{<}(\mathbf{M})$ are also nonempty.

Proposition 2.3. *(i) Let* $K_{\leqq}(\mathbf{M})$ *be acute. Then* $K^{s+}_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$.

(ii) Let $K_{\leqq}(\mathbf{M})$ *be acute. Then* $K^{s+}_{\leq}(\mathbf{M}) = K^{s+}_{\leqq}(\mathbf{M})$.

(iii) Let $\mathrm{cl}(K_{<}(\mathbf{M})) = K_{\leqq}(\mathbf{M})$. *Then* $K^{s+}_{<}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$.

Proof. (i) Since $K_{\leqq}(\mathbf{M}) \neq \emptyset$ is closed (by definition) and acute (by assumption), we have that $K^{s+}_{\leqq}(\mathbf{M}) = \text{int}(K^{+}_{\leqq}(\mathbf{M}))$ by Theorem 2.1(ii), Yu (1974). Further, since $\text{int}(K^{+}_{\leqq}(\mathbf{M})) \neq \emptyset$ by Theorem 2.1(i), Yu (1974), the interior and relative interior coincide (see formula (14), Dattorro 2015). Hence, Proposition 2.2(i) and Theorem 2.3.37, Greer (1984), yield the result.

(ii) By assumption, $K_{\leqq}(\mathbf{M})$ is acute, which implies that $K_{\leqq}(\mathbf{M})$ is pointed by Proposition 2.1. Hence, $\text{rank}(\mathbf{M}) = n$ by Theorem 3.1, Hunt et al. (2010), so that $\mathbf{x} = \mathbf{0}$ is the only solution to $\mathbf{M}\mathbf{x} = \mathbf{0}$ by Theorem 2.3, Bronson (1989). The latter implies that $K_{\leq}(\mathbf{M}) = K_{\leqq}(\mathbf{M}) \setminus \{\mathbf{0}\}$. Thus, by definition, we have that $K^{s+}_{\leqq}(\mathbf{M}) = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} > 0 \text{ for all } \mathbf{x} \in K_{\leqq}(\mathbf{M}) \setminus \{\mathbf{0}\}\}$, which is equal to $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} > 0 \text{ for all } \mathbf{x} \in K_{\leq}(\mathbf{M})\}$. Since $\mathbf{0} \notin K_{\leq}(\mathbf{M})$, the latter is equal to $K^{s+}_{\leq}(\mathbf{M})$ by definition, which yields $K^{s+}_{\leq}(\mathbf{M}) = K^{s+}_{\leqq}(\mathbf{M})$.

(iii) Since $K_{<}(\mathbf{M})$ is open, we know by Proposition 2.1.5(iv), Sawaragi et al. (1985), that $K^{s+}_{<}(\mathbf{M}) \cup \{\mathbf{0}\} = K^{+}_{<}(\mathbf{M})$, which implies that $K^{s+}_{<}(\mathbf{M}) = K^{+}_{<}(\mathbf{M}) \setminus \{\mathbf{0}\}$. As $K^{+}_{<}(\mathbf{M}) = K^{+}_{\leqq}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ by Proposition 2.2(ii) and $\boldsymbol{\lambda} = \mathbf{0}$ forces $\mathbf{x} = \mathbf{0}$, we obtain $K^{s+}_{<}(\mathbf{M}) = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0} : \mathbf{x} = -\mathbf{M}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$. Since $K_{<}(\mathbf{M}) \neq \emptyset$ (which is implied by our assumption), $\mathbf{M}\mathbf{x} < \mathbf{0}$ has a solution. Equivalently, by Gordan's Theorem (Mangasarian 1969), the system $-\mathbf{M}^T\boldsymbol{\lambda} = \mathbf{0}, \boldsymbol{\lambda} \geq \mathbf{0}$ has no solution, which yields the result. □

In the next section, Proposition 2.3 is used to a offer a new perspective on the efficiency of solutions to MOLPs.

2.2. On Multiobjective Linear Programming Efficiency

The results on efficiency we give in this section involve the cone of improving directions.

Definition 2.4. The *cone of improving directions* of MOLP (1.1) is defined to be $D_{\leq}(\mathbf{C}) := \{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}\mathbf{d} \leq \mathbf{0}\}$. The open and closed cones $D_{<}(\mathbf{C})$ and $D_{\leqq}(\mathbf{C})$ are defined accordingly.

Note that the cones of improving directions are equivalent to the cones $K_{<}(\mathbf{M})$, $K_{\leq}(\mathbf{M})$, and $K_{\leqq}(\mathbf{M})$. As such, their properties and related results are applicable. Using the cones of improving directions, we characterize the (weak) efficiency of solutions to MOLP (1.1).

Proposition 2.5. *Let* $\hat{\mathbf{x}} \in X$. *Then*

(i) $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{C})$ *if and only if* $(D_{\leq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap X = \emptyset$;

(ii) $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{C})$ *if* $(D_{\leqq}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap X = \{\hat{\mathbf{x}}\}$*;*

(iii) $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{C})$ *if and only if* $(D_{<}(\mathbf{C}) \oplus \{\hat{\mathbf{x}}\}) \cap X = \emptyset$*.*

Proof. Analogous to Proposition 1, Thoai (2012). □

In a separate theorem, Luc (2016) uses the normal cone to give a different necessary and sufficient condition for the (weak) efficiency of solutions to MOLP (1.1).

Theorem 2.6. *(Luc, 2016, Theorem 4.2.6) Let* $\hat{\mathbf{x}} \in X$*. Then*

(i) $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{C})$ *if and only if* $N_X(\hat{\mathbf{x}})$ *contains some vector* $-\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}$*;*

(ii) $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{C})$ *if and only if* $N_X(\hat{\mathbf{x}})$ *contains some vector* $-\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}$*.*

Given the cones of improving directions $D_{\leqq}(\mathbf{C}), D_{\leq}(\mathbf{C})$, and $D_{<}(\mathbf{C})$, we denote their polars by $D^{+}_{\leqq}(\mathbf{C}), D^{+}_{\leq}(\mathbf{C})$, and $D^{+}_{<}(\mathbf{C})$, and their strict polars by $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}), D^{\mathrm{s}+}_{\leq}(\mathbf{C})$, and $D^{\mathrm{s}+}_{<}(\mathbf{C})$, respectively. Under certain assumptions such as the acuteness or closure of the cones of improving directions, their polars and strict polars are given by Propositions 2.2 and 2.3, respectively. In Theorem 2.7, we provide a different point of view on Theorem 2.6 by considering these polars and strict polars.

Theorem 2.7. *Let* $\hat{\mathbf{x}} \in X$*.*

(i) *Assume* $D_{\leqq}(\mathbf{C})$ *is acute. Then* $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{C})$ *if and only if* $N_X(\hat{\mathbf{x}}) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}) \neq \emptyset$*.*

(ii) *Assume* $\mathrm{cl}(D_{<}(\mathbf{C})) = D_{\leqq}(\mathbf{C})$*. Then* $\hat{\mathbf{x}} \in \mathrm{wE}(X, \mathbf{C})$ *if and only if* $N_X(\hat{\mathbf{x}}) \cap D^{\mathrm{s}+}_{<}(\mathbf{C}) \neq \emptyset$*.*

Proof. (i) Since $D_{\leqq}(\mathbf{C})$ is acute, we know that $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ by Proposition 2.3(i). Hence, the result follows from Theorem 2.6(i).

(ii) Since $\mathrm{cl}(D_{<}(\mathbf{C})) = D_{\leqq}(\mathbf{C})$, we know that $D^{\mathrm{s}+}_{<}(\mathbf{C}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{C}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$ by Proposition 2.3(iii). Hence, the result follows from Theorem 2.6(ii). □

We note that Theorem 2.7(i) may be equivalently stated with $D^{\mathrm{s}+}_{\leq}(\mathbf{C})$ instead of $D^{\mathrm{s}+}_{\leqq}(\mathbf{C})$ since $D^{\mathrm{s}+}_{\leq}(\mathbf{C}) = D^{\mathrm{s}+}_{\leqq}(\mathbf{C})$ when $D_{\leqq}(\mathbf{C})$ is acute. In addition, even though $N_X(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in X$ and $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}) \neq \emptyset$ for all $\mathbf{C} \in \mathbb{R}^{p\times n}$ such that $D_{\leqq}(\mathbf{C})$ is acute (cf. Proposition 2.3(i)), this does not guarantee that $N_X(\mathbf{x}) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}) \neq \emptyset$ also. That being said, since $\mathrm{E}(X, \mathbf{C}) \neq \emptyset$ (as X is bounded), there exists $\hat{\mathbf{x}} \in X$ such that $N_X(\hat{\mathbf{x}}) \cap D^{s+}_{\leqq}(\mathbf{C}) \neq \emptyset$. Although Theorem 2.7 is weaker than Theorem 2.6 due to the additional assumptions about the cones of improving directions, the advantage of reframing Theorem 2.6 by means of $D^{s+}_{\leqq}(\mathbf{C})$ and $D^{s+}_{<}(\mathbf{C})$ is the added insight that we gain regarding HR(W)E solutions in Sections 3.1 and 3.2.

2.3. On Polars and Strict Polars of Unions of Convex Cones

Let $\mathbf{M}_1, \ldots, \mathbf{M}_\ell \in \mathbb{R}^{p\times n}$. We consider finite collections of the three types of cones obtained by means of the matrices $\mathbf{M}_k, k = 1, \ldots, \ell$, and derive algebraic formulas for the polar and strict polar cones of the unions of these collections. Our interest here in the unions of these convex cones is with regard to the relationship to the finite set of scenarios, which is evident in Section 3.1.

Proposition 2.8. *(i) The equality* $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^+ = \bigcap_{k=1}^{\ell} K^+_{\leqq}(\mathbf{M}_k)$ *holds.*

(ii) Let $\mathrm{cl}(K_<(\mathbf{M}_k)) = \mathrm{cl}(K_\leq(\mathbf{M}_k)) = K_{\leqq}(\mathbf{M}_k)$ *hold for all* $k = 1, \ldots, \ell$. *Then* $\left[\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)\right]^+ = \left[\bigcup_{k=1}^{\ell} K_\leq(\mathbf{M}_k)\right]^+ = \left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^+$.

Proof. (i) Follows directly from Proposition 2.1.6(i), Sawaragi et al. (1985).

(ii) By assumption, $K_<(\mathbf{M}_k) \neq \emptyset$ and $K_\leq(\mathbf{M}_k) \neq \emptyset$ for all $k = 1, \ldots, \ell$. Thus, Proposition 2.1.6(i), Sawaragi et al. (1985), Proposition 2.2(ii), and part (i), respectively, give the result. □

Since the polar obtained in Proposition 2.8 is the intersection of polyhedral convex cones, we may use, e.g., SageMath's `polyhedron base class`, The Sage Developers 2017, to compute its algebraic representation. Related to this intersection, we also have the following proposition.

Proposition 2.9. *(i) The polar* $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^+$ *is a polyhedral convex cone given by* $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *for some suitable matrix* $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$.

(ii) Let $\mathrm{cl}(K_<(\mathbf{M}_k)) = \mathrm{cl}(K_\leq(\mathbf{M}_k)) = K_{\leqq}(\mathbf{M}_k)$ *hold for all* $k = 1, \ldots, \ell$. *Then* $\left[\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)\right]^+ = \left[\bigcup_{k=1}^{\ell} K_\leq(\mathbf{M}_k)\right]^+$ *is a polyhedral convex cone given by* $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *for some suitable matrix* $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$.

Proof. (i) Since $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^+ = \bigcap_{k=1}^{\ell} K^+_{\leqq}(\mathbf{M}_k)$ by Proposition 2.8(i), and $K^+_{\leqq}(\mathbf{M}_k)$ is a polyhedral convex cone (in generator form) for each $k = 1, \ldots, \ell$, by Proposition 2.2(i), we conclude $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^+$ is also a polyhedral convex cone (see p. 84, Panik 1993). Thus, we may express it in generator form for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$.

(ii) Follows from part (i) and Proposition 2.8(ii). □

Remark 2.10. As previously noted, the polar of any nonempty cone is always nonempty as well since it is at least the origin. Hence, the matrix $\widetilde{\mathbf{M}}^T$ in Proposition 2.9 is guaranteed to exist, although we do not claim how to compute it. Moreover, in each instance above, the phrase "for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$" means "where the columns of $-\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$ are a finite set of generators of $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^+$". This notion is maintained throughout the paper.

We now determine algebraic formulas for the strict polar cones of the unions of collections of the three types of cones obtained by means of the matrices $\mathbf{M}_k, k = 1,\dots,\ell$. In order to do so, we need the following proposition that relates the intersection of the strict polars of two cones to the strict polar of the union of the two cones (cf. Proposition 2.1.6(i), Sawaragi et al. 1985).

Proposition 2.11. *Let $K_1, K_2 \subseteq \mathbb{R}^n$ be nonempty cones. Then $(K_1 \cup K_2)^{\mathrm{s}+} = K_1^{\mathrm{s}+} \cap K_2^{\mathrm{s}+}$.*

Proof. Let $\mathbf{z} \in K_1^{\mathrm{s}+} \cap K_2^{\mathrm{s}+}$, or equivalently, $\mathbf{z} \in K_1^{\mathrm{s}+}$ and $\mathbf{z} \in K_2^{\mathrm{s}+}$. By definition, $\mathbf{x}^T\mathbf{z} > 0$ for any $\mathbf{x} \in K_1 \setminus \{\mathbf{0}\}$ and $\mathbf{x}^T\mathbf{z} > 0$ for any $\mathbf{x} \in K_2 \setminus \{\mathbf{0}\}$. Equivalently, $\mathbf{x}^T\mathbf{z} > 0$ for any $\mathbf{x} \in (K_1 \cup K_2) \setminus \{\mathbf{0}\}$, i.e., $\mathbf{z} \in (K_1 \cup K_2)^{\mathrm{s}+}$ as desired. □

We next extend Proposition 2.11 to the union of finite collections of the three convex cones under investigation.

Proposition 2.12. *(i) The equality $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^{\mathrm{s}+} = \bigcap_{k=1}^{\ell} K_{\leqq}^{\mathrm{s}+}(\mathbf{M}_k)$ holds.*

(ii) Let $K_{\leq}(\mathbf{M}_k) \neq \emptyset$ and $K_{\leqq}(\mathbf{M}_k)$ be acute for all $k = 1,\dots,\ell$. Then $\left[\bigcup_{k=1}^{\ell} K_{\leq}(\mathbf{M}_k)\right]^{\mathrm{s}+} = \left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^{\mathrm{s}+}$.

(iii) Let $K_{<}(\mathbf{M}_k) \neq \emptyset$ for all $k = 1,\dots,\ell$. Then $\left[\bigcup_{k=1}^{\ell} K_{<}(\mathbf{M}_k)\right]^{\mathrm{s}+} = \bigcap_{k=1}^{\ell} K_{<}^{\mathrm{s}+}(\mathbf{M}_k)$.

Proof. (i) & (iii) Follow directly from Proposition 2.11.

(ii) Follows from part (i), Proposition 2.3(ii), and Proposition 2.11, respectively. □

Under an acuteness condition and a condition on the closure of $K_{<}(\mathbf{M}_k)$ (similar to that used in Proposition 2.3), we obtain algebraic formulas for the strict polars given in Proposition 2.12.

Theorem 2.13. *(i) Let $\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)$ be acute. Then $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^{\mathrm{s}+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$.*

(ii) Let $\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)$ *be acute. Then* $\left[\bigcup_{k=1}^{\ell} K_{\leq}(\mathbf{M}_k)\right]^{\mathrm{s}+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ *for some suitable matrix* $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$.

(iii) Let $\mathrm{cl}(K_<(\mathbf{M}_k)) = K_{\leqq}(\mathbf{M}_k)$ *hold for all* $k = 1, \ldots, \ell$*. Then* $\left[\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)\right]^{\mathrm{s}+} = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0} : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$ *for some suitable matrix* $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$*. Moreover, if* $K_<(\widetilde{\mathbf{M}}) \neq \emptyset$*, then* $\left[\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)\right]^{\mathrm{s}+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \mathbf{0}\}$.

Proof. (i) Since $\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k) \neq \emptyset$ is acute and closed (as a finite union of closed sets is closed), Theorem 2.1, Yu (1974), and Proposition 2.9(i) give $\left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right]^{\mathrm{s}+} = \mathrm{int}(\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}) \neq \emptyset$ for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$. Thus, applying Theorem 2.3.37, Greer (1984), we obtain the result.

(ii) Since $\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)$ is acute, $K_{\leqq}(\mathbf{M}_k)$ is also acute for all $k = 1, \ldots, \ell$. Hence, we obtain $\bigcup_{k=1}^{\ell} K_{\leq}(\mathbf{M}_k) = \left[\bigcup_{k=1}^{\ell} K_{\leqq}(\mathbf{M}_k)\right] \setminus \{\mathbf{0}\}$ and the desired result as in Proposition 2.3(ii).

(iii) Since $\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)$ is open (as an arbitrary union of open sets is open), we know by Proposition 2.1.5(iv), Sawaragi et al. (1985), that $\left[\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)\right]^{\mathrm{s}+} = \left[\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)\right]^{+} \setminus \{\mathbf{0}\}$. That is, $\left[\bigcup_{k=1}^{\ell} K_<(\mathbf{M}_k)\right]^{\mathrm{s}+} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{M}}^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\} \setminus \{\mathbf{0}\}$ for some suitable matrix $\widetilde{\mathbf{M}}^T \in \mathbb{R}^{n\times\tilde{p}}$ by Proposition 2.9(ii). Since $\boldsymbol{\lambda} = \mathbf{0}$ forces $\mathbf{x} = \mathbf{0}$, the first part of the result follows. For the second part, let $K_<(\widetilde{\mathbf{M}}) \neq \emptyset$. Thus, Gordan's Theorem (Mangasarian 1969) yields the result. □

As with Proposition 2.3, the strict polars derived in Theorem 2.13 are clearly nonempty. In the next section, Proposition 2.12 and Theorem 2.13 lead toward a sufficient condition for highly robust (weak) efficiency and a lower bound set on the HRE set, respectively.

3. Regarding the HRE Set

In this section, we explore properties of the HRE set such as those that we extend from deterministic efficiency, as well as those specific to UMOLPs and the definition of highly robust efficiency. The former are examined directly below, while the latter, including a characterization of the HRE set and bound sets on the HRE set, are presented in the subsequent subsections. As acuteness emerges as an important element in the course of this analysis, we address this property in more detail at the end of this section.

Various properties of the efficient set of MOLP (1.1) are known in the literature. We examine how some of these properties extend from efficient solutions in the deterministic case to HRE solutions in the uncertain case. In particular, we provide five properties of the efficient set of MOLP (1.1) that directly extend to the HRE set of MOLP(U), and one that does not. In the following, a set that is not *disconnected* (see p. 78, Carothers 2000) is *connected*.

Proposition 3.1. *(i)* $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *is closed.*

(ii) $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *is not necessarily convex.*

(iii) If $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$*, then it is either the entire set* X *or on the boundary of* X*.*

(iv) If $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$*, then there exists an HRE extreme point.*

(v) If $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$ *and a point on the relative interior of a face of* X *is HRE, then so is the entire face.*

(vi) $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *is not necessarily connected.*

Proof. (i) Since MOLP($\mathbf{u}$) is a deterministic MOLP for each $\mathbf{u} \in U$, we have that $\mathrm{E}(X, \mathbf{C}(\mathbf{u}))$ is closed for each $\mathbf{u} \in U$ by Theorem 4.1.20, Luc (2016). Hence, as an arbitrary intersection of closed sets is closed, the result follows by Definition 1.1.

(ii)-(v) Similarly, (ii)-(v) follow by Definition 1.1 and Example 7.24, Lemma 7.17, Lemma 7.1, and Theorem 7.20, Ehrgott (2005), respectively.

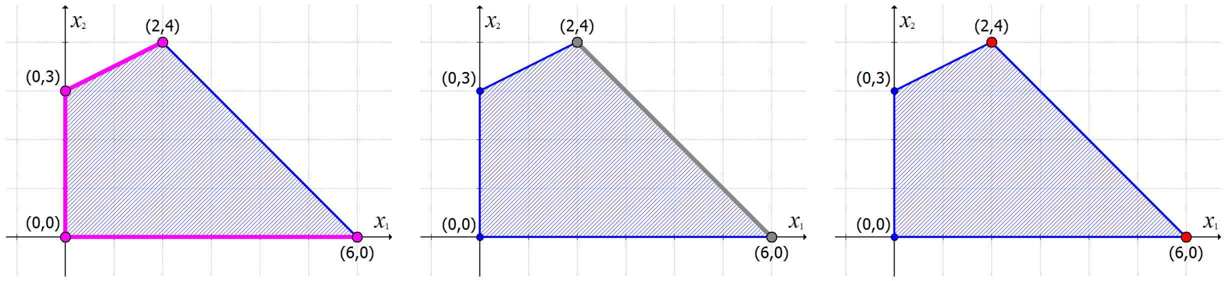
(vi) As $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ is the intersection of possibly nonconvex sets, it may be disconnected. □

Although $\mathrm{E}(X, \mathbf{C})$ is connected when X is bounded (see Theorem 6.5.4, Ehrgott 2005) and the first five properties in Proposition 3.1 extend directly from the deterministic to uncertain setting, the same is not true of connectedness. As an illustration, consider the following example.

Example 3.2. Consider the following UMOLP:

$$\left\{ \min_{\mathbf{x} \in X_1} \begin{bmatrix} 3u_{11} & -9u_{12} \\ -u_{21} & 9u_{22} \end{bmatrix} \mathbf{x} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{3.1}$$

where $U_1 = \{(1,1)\}, U_2 = \{(1,1), (2, -1/9)\}$, and X_1 is given by (1.6). We observe that the HRE set is disconnected, as shown in Figure 2c.



(a) The efficient set under scenario $(1, 1, 1, 1)$ (purple)

(b) The efficient set under scenario $(1, 1, 2, -1/9)$ (grey)

(c) The HRE set of UMOLP (3.1) (red)

Figure 2: Efficient and HRE points for Example 3.2

3.1. Characterization of the HRE Set

Similarly to properties of the HRE set, we extend known results about the efficient set of MOLP (1.1) that use convex cones (such as the cone of improving directions and the normal cone) to those regarding the HRE set in the uncertain setting.

We first examine the cone of improving directions. As each instance of MOLP(U) is a deterministic MOLP, we may denote the cones of improving directions of MOLP($\mathbf{u}$) for each scenario $\mathbf{u} \in U$ as in the deterministic setting, where $\mathbf{C}$ is replaced by $\mathbf{C}(\mathbf{u})$. In addition, we may define the cones of improving directions of MOLP(U) by accounting for the improving directions associated with every scenario $\mathbf{u} \in U$.

Definition 3.3. The *cone of improving directions* of MOLP(U) is defined to be $D_{\leq}(\mathbf{C}(\mathbf{u}), U) := \bigcup_{\mathbf{u}\in U} D_{\leq}(\mathbf{C}(\mathbf{u}))$. The open and closed cones $D_{<}(\mathbf{C}(\mathbf{u}), U)$ and $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ are defined accordingly.

In the deterministic setting, the cones of improving directions of MOLP (1.1) may be used to characterize the (weak) efficiency of solutions as in Proposition 2.5. Analogously to the deterministic case, we may characterize the highly robust (weak) efficiency of solutions to MOLP(U) using the cones of improving directions given in Definition 3.3.

Theorem 3.4. *Let $\mathbf{x}^* \in X$. Then*

(i) $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *if and only if* $(D_{\leq}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap X = \emptyset$*;*

(ii) $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *if* $(D_{\leqq}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap X = \{\mathbf{x}^*\}$*;*

(iii) $\mathbf{x}^* \in \mathrm{wE}(X, \mathbf{C}(\mathbf{u}), U)$ *if and only if* $(D_<(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap X = \emptyset$.

Proof. (i) Since $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}))$ if and only if $(D_\leq(\mathbf{C}(\mathbf{u})) \oplus \{\mathbf{x}^*\}) \cap X = \emptyset$ by Proposition 2.5(i), we can likewise say that $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ if and only if $(D_\leq(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap X = \emptyset$ for all $i = 1, \ldots, s$. Equivalently, $\big[(D_\leq(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}) \cap X\big] \cup \cdots \cup \big[(D_\leq(\mathbf{C}(\mathbf{u}^s)) \oplus \{\mathbf{x}^*\}) \cap X\big] = \emptyset$, i.e., $\big[(D_\leq(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}) \cup \cdots \cup (D_\leq(\mathbf{C}(\mathbf{u}^s)) \oplus \{\mathbf{x}^*\})\big] \cap X = \emptyset$ by the Distributive Law of Intersections. Moreover, by formula 1-5-5, Matheron (1975), we equivalently obtain $\big[\big(\bigcup_{\mathbf{u}\in U} D_\leq(\mathbf{C}(\mathbf{u}))\big) \oplus \{\mathbf{x}^*\}\big] \cap X = \emptyset$. Applying Definition 3.3, we obtain the result.

(ii) Let $(D_{\leqq}(\mathbf{C}(\mathbf{u}), U) \oplus \{\mathbf{x}^*\}) \cap X = \{\mathbf{x}^*\}$. By definition, $\big[(D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cup \cdots \cup D_{\leqq}(\mathbf{C}(\mathbf{u}^s))) \oplus \{\mathbf{x}^*\}\big] \cap X = \{\mathbf{x}^*\}$. Equivalently, by formula 1-5-5, Matheron (1975), and the Distributive Law of Intersections, respectively, we have $\big[(D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}) \cap X\big] \cup \cdots \cup \big[(D_{\leqq}(\mathbf{C}(\mathbf{u}^s)) \oplus \{\mathbf{x}^*\}) \cap X\big] = \{\mathbf{x}^*\}$. That is, either $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap X = \{\mathbf{x}^*\}$ or $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap X = \emptyset$ for each $i = 1, \ldots, s$, with at least one equal to $\{\mathbf{x}^*\}$. However, it is clear that $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap X \neq \emptyset, i \in \{1, \ldots, s\}$, since $D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}$ must contain at least $\mathbf{x}^* \in X$. Hence, $(D_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \oplus \{\mathbf{x}^*\}) \cap X = \{\mathbf{x}^*\}$ for all $i = 1, \ldots, s$, which implies that $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}^i))$ for all $i = 1, \ldots, s$, by Proposition 2.5(ii). Thus, $\mathbf{x}^*$ is HRE by definition.

(iii) Follows similarly to the proof of part (i). □

Remark 3.5. It is worth noting that if $D_\leq(\mathbf{C}(\mathbf{u}), U) = \emptyset$, then $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) = X$ since $\emptyset \oplus \{\mathbf{x}^*\} = \emptyset$ (see p. 16, Matheron 1975) so that the condition in (i) holds trivially for all $\mathbf{x}^* \in X$. Similarly, if $D_<(\mathbf{C}(\mathbf{u}), U) = \emptyset$, then $\mathrm{wE}(X, \mathbf{C}(\mathbf{u}), U) = X$. Moreover, if $D_\leq(\mathbf{C}(\mathbf{u}), U) = \mathbb{R}^n$, then $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) = \emptyset$ since $\mathbb{R}^n \oplus \{\mathbf{x}^*\} = \mathbb{R}^n$ so that the condition in (i) does not hold for any $\mathbf{x}^* \in X$. Likewise, if $D_<(\mathbf{C}(\mathbf{u}), U) = \mathbb{R}^n$, then $\mathrm{wE}(X, \mathbf{C}(\mathbf{u}), U) = \emptyset$.

We may also extend Theorems 2.6 and 2.7, which use the normal cone, from the deterministic to uncertain setting. As mentioned earlier, by reframing the theorem due to Luc in the context of the strict polars of the cones of improving directions, we achieve a different perspective that leads to further insight in the form of conditions on highly robust (weak) efficiency. Recasting this theorem also allows us to exploit properties of cones. To this end, as each instance MOLP$(\mathbf{u})$ of MOLP(U) is a deterministic MOLP, the strict polars of the cones of improving directions of MOLP$(\mathbf{u})$ are given by Proposition 2.3, where $\mathbf{M}$ is replaced by $\mathbf{C}(\mathbf{u})$.

Remark 3.6. (i) We extend Theorem 2.6 as follows. For a solution $\mathbf{x}^* \in X$, it is HR(W)E if and only if $N_X(\mathbf{x}^*)$ contains some vector $-\mathbf{C}(\mathbf{u})^T\boldsymbol{\lambda}, \boldsymbol{\lambda}\ (\geq) > \mathbf{0}$, for all $\mathbf{u} \in U$. It is worth noting that if $-\mathbf{C}(\mathbf{u})^T\boldsymbol{\lambda} = \mathbf{0}$ for some $\mathbf{u} \in U$ and some $\boldsymbol{\lambda} > \mathbf{0}$, then the entire feasible set is efficient in that scenario since $N_X(\mathbf{x}^*)$ necessarily contains $\mathbf{0}$. Similarly, if for all $\mathbf{u} \in U$ there exists a $\boldsymbol{\lambda} > \mathbf{0}$ such that $-\mathbf{C}(\mathbf{u})^T\boldsymbol{\lambda} = \mathbf{0}$, then the entire feasible set is in fact HRE. (The same line of thought may be followed for $\boldsymbol{\lambda} \geq \mathbf{0}$ and the HRWE set.)

(ii) Similarly, we extend Theorem 2.7 (under the same assumptions, but for all $\mathbf{u} \in U$) by saying that $\mathbf{x}^* \in X$ is HR(W)E if and only if $(N_X(\mathbf{x}^*) \cap D^{s+}_{<}(\mathbf{C}(\mathbf{u})) \neq \emptyset)$ $N_X(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$. As in Theorem 2.7, we may equivalently use $D^{s+}_{\leq}(\mathbf{C}(\mathbf{u}))$ since $D^{s+}_{\leq}(\mathbf{C}(\mathbf{u})) = D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}))$ when $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute by Proposition 2.3(ii). Moreover, as we need $N_X(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$, it is important to know when $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ since if it is not, the result never holds. (We are only concerned with the nonemptiness of $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}))$ since $N_X(\mathbf{x}^*) \neq \emptyset$.) To this end, it is clear that $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ by Proposition 2.3(i).

In order to obtain a result that does not require checking the necessary and sufficient conditions of Theorems 2.6 and 2.7 for every scenario $\mathbf{u} \in U$, we use the strict polars of the cones of improving directions of MOLP(U) (cf. Proposition 2.12, where $\mathbf{M}_k$ is replaced by $\mathbf{C}(\mathbf{u})$). Given the cones of improving directions $D_{\leqq}(\mathbf{C}(\mathbf{u}), U), D_{\leq}(\mathbf{C}(\mathbf{u}), U)$, and $D_{<}(\mathbf{C}(\mathbf{u}), U)$ of MOLP(U), we denote their strict polars by $D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U), D^{s+}_{\leq}(\mathbf{C}(\mathbf{u}), U)$, and $D^{s+}_{<}(\mathbf{C}(\mathbf{u}), U)$, respectively.

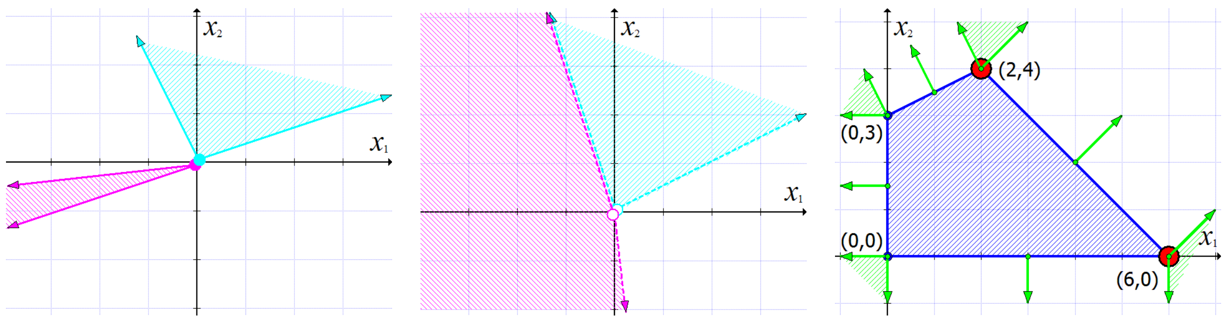
Theorem 3.7. *Let* $\mathbf{x}^* \in X$*.*

(i) Let $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *be acute for all* $\mathbf{u} \in U$*. If* $N_X(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$*, then* $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$*.*

(ii) Let $\mathrm{cl}(D_{<}(\mathbf{C}(\mathbf{u}))) = D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *for all* $\mathbf{u} \in U$*. If* $N_X(\mathbf{x}^*) \cap D^{s+}_{<}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$*, then* $\mathbf{x}^* \in \mathrm{wE}(X, \mathbf{C}(\mathbf{u}), U)$*.*

Proof. (i) Let $N_X(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$. Equivalently, by Proposition 2.12(i), $N_X(\mathbf{x}^*) \cap \bigcap_{\mathbf{u} \in U} D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$. That is, $\left[N_X(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^1))\right] \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^2)) \cap \cdots \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^s)) \neq \emptyset$ by the Associative Law of Intersections. Accordingly, the associative law yields $N_X(\mathbf{x}^*) \cap D^{s+}_{\leqq}(\mathbf{C}(\mathbf{u}^i)) \neq \emptyset$ for all $i = 1, \ldots, s$. Thus, the result follows from Theorem 2.7(i).

(ii) Follows similarly to the proof of part (i), where $\mathrm{cl}(D_{<}(\mathbf{C}(\mathbf{u}))) = D_{\leqq}(\mathbf{C}(\mathbf{u}))$ implies that $D_{<}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ so that we may use Proposition 2.12(iii). □



(a) $D_{\leqq}(\mathbf{C}(\mathbf{u}^1))$ (purple) and $D_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ (teal)

(b) $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^1))$ (purple) and $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ (teal)

(c) The HRE set (red) and normal cones (green)

Figure 3: Normal cones to X_1, and the closed cones of improving directions and their strict polars for Example 3.9

Remark 3.8. As in Remark 3.6(ii), it is of interest to know when $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$ since if it is not, then $N_X(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$ never holds. To this end, since $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is a closed cone, we know that $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$ when $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute due to Theorem 2.1, Yu (1974). Moreover, with the additional assumption that $D_{\leq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ (which is needed for Proposition 2.12(ii)), we may rewrite Theorem 3.7(i) using $D^{\mathrm{s}+}_{\leq}(\mathbf{C}(\mathbf{u}), U)$.

For an illustration of Theorem 3.7(i), as well as the extension of Theorem 2.7(i) described in Remark 3.6(ii), consider the following example.

Example 3.9. Consider UMOLP (3.1) with $U_1 = \{(1, 1)\}$ and $U_2 = \{(1, 1), (2, -1/9)\}$ as in Example 3.2. We have two scenarios $\mathbf{u}^1 = (1, 1, 1, 1)$ and $\mathbf{u}^2 = (1, 1, 2, -1/9)$. The closed cones of improving directions $D(\mathbf{C}(\mathbf{u}^1))$ and $D(\mathbf{C}(\mathbf{u}^2))$ are shown in Figure 3a, while their strict polars are shown in Figure 3b. Since $D_{\leqq}(\mathbf{C}(\mathbf{u}^i))$ is acute for $i = 1, 2$, the assumptions of Theorems 2.7(i) (for each $\mathbf{u} \in U$) and 3.7(i) hold. As illustrated in Figure 3, the only points at which Theorem 2.7(i) holds for each $\mathbf{u} \in U$ are the two HRE points $(2, 4)$ and $(6, 0)$. However, as $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^2)) = \emptyset$ (clearly shown in Figure 3b), the sufficient condition of Theorem 3.7(i) does not hold (trivially) at either HRE point, so we are unable to identify either point via this theorem.

Similarly, using the union of strict polars rather than the intersection, we obtain a necessary condition for highly robust (weak) efficiency.

Theorem 3.10. *Let* $\mathbf{x}^* \in X$*.*

(i) Assume $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ *is acute for all* $\mathbf{u} \in U$*. If* $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$*, then*

$N_X(\mathbf{x}^*) \cap \bigcup_{\mathbf{u}\in U} D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$.

(ii) Assume $\mathrm{cl}(D_<(\mathbf{C}(\mathbf{u}))) = D_{\leq}(\mathbf{C}(\mathbf{u}))$ *for all* $\mathbf{u} \in U$. *If* $\mathbf{x}^* \in \mathrm{wE}(X, \mathbf{C}(\mathbf{u}), U)$, *then* $N_X(\mathbf{x}^*) \cap \bigcup_{\mathbf{u}\in U} D^{\mathrm{s}+}_{<}(\mathbf{C}(\mathbf{u})) \neq \emptyset$.

Proof. (i) Let $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$. Equivalently, $N_X(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ for all $\mathbf{u} \in U$ by Theorem 2.7(i). Since $N_X(\mathbf{x}^*) \cap \bigcup_{\mathbf{u}\in U} D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u})) = \left[N_X(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^1))\right] \cup \cdots \cup \left[N_X(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}^s))\right]$ by the Distributive Law of Intersections, the result follows.

(ii) Follows similarly to the proof of part (i). □

It is important to note that since Theorem 2.7, which is both necessary and sufficient, is split into two separate theorems, Theorems 3.7 and 3.10, one that is sufficient and the other that is necessary, respectively, we lose the strength of the original theorem. This is supported by Example 3.9 in which applying Theorem 2.7(i) for each scenario yields the entire HRE set, while applying Theorem 3.7(i) does not yield any HRE solutions yet the entire boundary satisfies the consequent of Theorem 3.10(i) even though the entire boundary is not HRE.

3.2. Bound Sets and a Robust Counterpart

In robust optimization, an RC, which is a deterministic (scalar or vector) optimization problem associated with the original uncertain optimization problem whose solutions are the desired robust solutions, is commonly used. The solution set of an RC may be interpreted as both an upper and lower bound set on the set of robust solutions to the original uncertain problem. Working toward an RC to obtain HRE solutions to MOLP(U), in this section, we develop several bound sets on the HRE set, and then present an RC for a special class of UMOLPs.

First, we know that, in general, the efficient set of any instance MOLP($\mathbf{u}$) is an upper bound set on the HRE set of MOLP(U).

Proposition 3.11. *The containment* $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(X, \mathbf{C}(\mathbf{u}))$ *holds for every* $\mathbf{u} \in U$.

Proof. Immediate since $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u}\in U} \mathrm{E}(X, \mathbf{C}(\mathbf{u}))$. □

Another upper bound set on the HRE set is given by the efficient set of the so-called *all-in-one problem.* The all-in-one problem, denoted AIOP(U), is given by $\min_{\mathbf{x}\in X} \mathbf{C}(U)\mathbf{x}$, where $\mathbf{C}(U) := \begin{bmatrix}\mathbf{C}(\mathbf{u}^1) & \cdots & \mathbf{C}(\mathbf{u}^s)\end{bmatrix}^T \in \mathbb{R}^{ps\times n}$ is a deterministic cost matrix given U. Immediately, since

AIOP(U) is a deterministic MOLP whose efficient solutions are determined by *ps* criteria, we know that HRE solutions to MOLP(U) are at least weakly efficient solutions to AIOP(U) based on Proposition 1, Engau & Wiecek (2008). Even more, as shown in Wiecek & Dranichak (2016), the HRE set is contained in the efficient set of AIOP(U), which is denoted $\mathrm{E}(X, \mathbf{C}(U))$.

Proposition 3.12. *(Wiecek & Dranichak, 2016, Proposition 8) The containment* $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(X, \mathbf{C}(U))$ *holds.*

Third, for a special class of UMOLPs, we may obtain an additional upper bound set. In order to obtain the upper bound set, we need the following lemma.

Lemma 3.13. *Let* $\min_{\mathbf{x}\in X} \mathbf{C}_1\mathbf{x}$ *and* $\min_{\mathbf{x}\in X} \mathbf{C}_2\mathbf{x}$ *be given. If* $D_{\leq}(\mathbf{C}_1) \subseteq D_{\leq}(\mathbf{C}_2)$*, then* $\mathrm{E}(X, \mathbf{C}_2) \subseteq \mathrm{E}(X, \mathbf{C}_1)$*.*

Proof. Suppose $D_{\leq}(\mathbf{C}_1) \subseteq D_{\leq}(\mathbf{C}_2)$, and assume for the sake of contradiction that $\mathrm{E}(X, \mathbf{C}_2) \nsubseteq \mathrm{E}(X, \mathbf{C}_1)$, i.e., there exists $\hat{\mathbf{x}} \in \mathrm{E}(X, \mathbf{C}_2)$ such that $\hat{\mathbf{x}} \notin \mathrm{E}(X, \mathbf{C}_1)$. The former implies that $D_{\leq}(\mathbf{C}_1) \oplus \{\mathbf{x}\} \subseteq D_{\leq}(\mathbf{C}_2) \oplus \{\mathbf{x}\}$ for all $\mathbf{x} \in X$, while the latter yields $[D_{\leq}(\mathbf{C}_2) \oplus \{\hat{\mathbf{x}}\}] \cap X = \emptyset$, but $[D_{\leq}(\mathbf{C}_1) \oplus \{\hat{\mathbf{x}}\}] \cap X \neq \emptyset$ by Proposition 2.5(i). Hence, $\emptyset \neq [D_{\leq}(\mathbf{C}_1) \oplus \{\hat{\mathbf{x}}\}] \cap X \subseteq [D_{\leq}(\mathbf{C}_2) \oplus \{\hat{\mathbf{x}}\}] \cap X = \emptyset$, which is a contradiction. Thus, it must be that $\mathrm{E}(X, \mathbf{C}_2) \subseteq \mathrm{E}(X, \mathbf{C}_1)$ as desired. □

Using this lemma, we prove the following upper bound set on the HRE set, which is an extension of Proposition 3.1, Bitran (1980).

Theorem 3.14. *Suppose each column of* $\mathbf{C}(\mathbf{u})$ *is either nonnegative for all* $\mathbf{u} \in U$ *or nonpositive for all* $\mathbf{u} \in U$ *with no column all 0. Let* $\mathbf{I}$ *be the diagonal matrix with a 1 corresponding to the nonnegative columns of* $\mathbf{C}(\mathbf{u})$ *and a* -1 *for the nonpositive columns. The containment* $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \subseteq \mathrm{E}(X, \mathbf{I})$ *holds.*

Proof. Let I and J be subsets of the index set $\{1, \ldots, n\}$ for which the columns of $\mathbf{C}(\mathbf{u})$ are nonnegative for all $\mathbf{u} \in U$ and nonpositive for all $\mathbf{u} \in U$, respectively. The cones of improving directions associated with $\min_{\mathbf{x}\in X} \mathbf{I}\mathbf{x}$ and an instance MOLP($\mathbf{u}$) are given by $D_{\leq}(\mathbf{I}) = \{\mathbf{d} \in \mathbb{R}^n : d_i \leq 0, i \in I, d_j \geq 0, j \in J, \text{ at least one strict}\}$, and $D_{\leq}(\mathbf{C}(\mathbf{u})) = \{\mathbf{d} \in \mathbb{R}^n : c_{11}u_{11}d_1 + \cdots + c_{1n}u_{1n}d_n \leq 0, \ldots, c_{p1}u_{p1}d_1 + \cdots + c_{pn}u_{pn}d_n \leq 0, \text{ at least one strict}\}$, respectively, where $c_{ki}u_{ki} \geq 0$ for all $k = 1, \ldots, p, i \in I$, and $c_{kj}u_{kj} \leq 0$ for all $k = 1, \ldots, p, j \in J$, by assumption. If $\mathbf{d} \in D_{\leq}(\mathbf{I})$, then $d_i \leq 0, i \in I$, and $d_j \geq 0, j \in J$, with at least one strict. Since $c_{ki}u_{ki} \geq 0$ for all $i \in I$ and $c_{kj}u_{kj} \leq 0$ for all $j \in J$, clearly $\mathbf{d} \in D_{\leq}(\mathbf{C}(\mathbf{u}))$ also (which is not true, however, without the assumption that no column is entirely 0). Hence, $D_{\leq}(\mathbf{I}) \subseteq D_{\leq}(\mathbf{C}(\mathbf{u}))$ for all $\mathbf{u} \in U$, which implies that $\mathrm{E}(X, \mathbf{C}(\mathbf{u})) \subseteq \mathrm{E}(X, \mathbf{I})$ for all $\mathbf{u} \in U$ by Lemma 3.13. Thus, $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u} \in U} \mathrm{E}(X, \mathbf{C}(\mathbf{u})) \subseteq \mathrm{E}(X, \mathbf{I})$ as desired. □

The assumptions in Theorem 3.14, although conspicuous, are realistic in practice. For example, problems in bank balance sheet management, portfolio management, and knapsack packing generally satisfy these assumptions.

For MOLP(U) in general, we can obtain another bound set (either upper or lower) with a theorem similar to Lemma 3.13. As the theorem involves two different uncertainty sets, it can also be used to provide additional information to decision makers by presenting the effects of adding or removing scenarios from a given uncertainty set.

Theorem 3.15. *Let* $\{\min_{\mathbf{x} \in X} \mathbf{C}(\mathbf{u})\mathbf{x}\}_{\mathbf{u} \in U'}$ *and* $\{\min_{\mathbf{x} \in X} \mathbf{C}(\mathbf{u})\mathbf{x}\}_{\mathbf{u} \in U''}$ *be given. If* $D_{\leq}(\mathbf{C}(\mathbf{u}), U') \subseteq D_{\leq}(\mathbf{C}(\mathbf{u}), U'')$, *then* $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U'') \subseteq \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U')$.

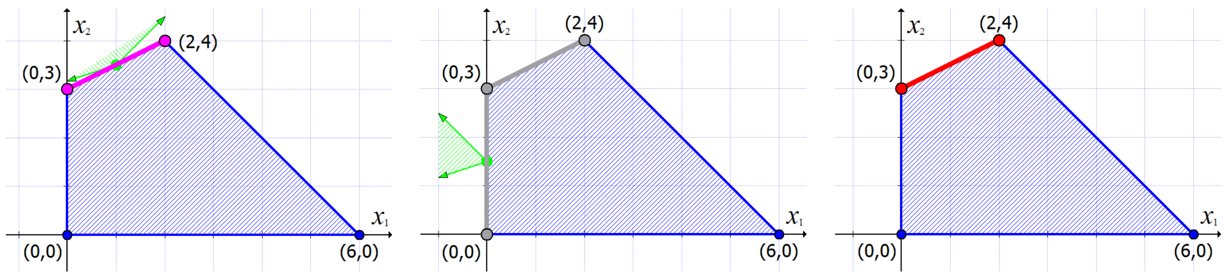
Proof. Follows similarly to the proof of Lemma 3.13, except that Theorem 3.4(i) is used instead of Proposition 2.5(i). □

In order to obtain a lower bound set on the HRE set, we utilize the sufficient condition of Theorem 3.7(i).

Theorem 3.16. *Assume* $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ *is acute. Then* $\mathrm{E}(X, \widetilde{\mathbf{C}}) \subseteq \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *for some suitable matrix* $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n \times \tilde{p}}$.

Proof. We have that $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{C}}^T \boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$ for some suitable matrix $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n \times \tilde{p}}$ by Theorem 2.13(i). Hence, we may write $D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) = D^{\mathrm{s}+}_{\leqq}(\widetilde{\mathbf{C}})$, where $D_{\leqq}(\widetilde{\mathbf{C}})$ is associated with the deterministic MOLP given by $\min_{\mathbf{x} \in X} \widetilde{\mathbf{C}}\mathbf{x}$. Equivalently, for $\mathbf{x}^* \in \mathrm{E}(X, \widetilde{\mathbf{C}})$, we have that $N_X(\mathbf{x}^*) \cap D^{\mathrm{s}+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$ by Theorem 2.7(i). Consequently, since $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ being acute implies that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute for all $\mathbf{u} \in U$, we have that $\mathbf{x}^* \in \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ also by Theorem 3.7(i). Therefore, $\mathrm{E}(X, \widetilde{\mathbf{C}}) \subseteq \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ as desired. □

With regard to obtaining an MOLP whose efficient set is equal to the HRE set rather than a bound set as in the above results, we need an assumption that is stronger than those made previously.



(a) $\mathrm{E}(X_1, \mathbf{C}(\mathbf{u}^1))$ (purple) and $D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \oplus \{\mathbf{x}^*\}$ (green)

(b) $\mathrm{E}(X_1, \mathbf{C}(\mathbf{u}^2))$ (grey) and $D_{\leqq}(\mathbf{C}(\mathbf{u}^2)) \oplus \{\mathbf{x}^*\}$ (green)

(c) The HRE set of UMOLP (3.3) (red)

Figure 4: Efficient and HRE points for Example 3.18

Theorem 3.17. *Assume $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is a polyhedral convex cone. Then* $\mathrm{E}(X, \overline{\mathbf{C}}) = \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *for some suitable matrix $\overline{\mathbf{C}} \in \mathbb{R}^{\bar{p} \times n}$.*

Proof. By assumption, we may write $D_{\leqq}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{d} \in \mathbb{R}^n : \overline{\mathbf{C}}\mathbf{d} \leqq \mathbf{0}\}$ for some suitable matrix $\overline{\mathbf{C}} \in \mathbb{R}^{\bar{p} \times n}$. Here, the suitability of $\overline{\mathbf{C}}$ means that the rows of $\overline{\mathbf{C}}$ are the normals to the generating hyperplanes whose half-spaces form $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$. Hence, $D_{\leq}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{d} \in \mathbb{R}^n : \overline{\mathbf{C}}\mathbf{d} \leq \mathbf{0}\} = D_{\leq}(\overline{\mathbf{C}})$, which is the cone of improving directions of the deterministic MOLP given by $\min_{\mathbf{x} \in X} \overline{\mathbf{C}}\mathbf{x}$. Since $D_{\leq}(\mathbf{C}(\mathbf{u}), U)$ is the cone of improving directions of both MOLP(U) and $\min_{\mathbf{x} \in X} \overline{\mathbf{C}}\mathbf{x}$, we obtain $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) = \mathrm{E}(X, \overline{\mathbf{C}})$ by Proposition 2.5(i) and Theorem 3.4(i). □

The deterministic MOLP implied by Theorem 3.17, which is given by

$$\min_{\mathbf{x} \in X} \quad \overline{\mathbf{C}}\mathbf{x}, \tag{3.2}$$

is an RC of MOLP(U) since a solution to MOLP(U) is HRE if and only if it is an efficient solution to MOLP (3.2). For an illustration of Theorems 3.16 and 3.17, including computing the associated RC, consider the following example.

Example 3.18. Consider the following UMOLP:

$$\left\{ \min_{\mathbf{x} \in X_1} \quad \begin{bmatrix} u_{11} & -3u_{12} \\ u_{21} & u_{22} \end{bmatrix} \mathbf{x} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2,} \tag{3.3}$$

where $U_1 = \{(1, 1)\}$ and $U_2 = \{(1, -1), (1, 1)\}$. For scenarios $\mathbf{u}^1 = (1, 1, 1, -1)$ and $\mathbf{u}^2 = (1, 1, 1, 1)$, it is clear that $D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cup D_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ is an acute polyhedral convex cone (as the union is simply

$D_{\leqq}(\mathbf{C}(\mathbf{u}^1)))$, which is shown in Figure 4. Hence, we have that the cost matrix of the RC is

$$\overline{\mathbf{C}} = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix},$$

and observe that $\widetilde{\mathbf{C}} = \overline{\mathbf{C}}$ in this example since $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is both acute and polyhedral convex.

As a final result in this section, we use Theorem 3.16 or 3.17 in order to show the conditions under which the HRE set is nonempty.

Corollary 3.19. *Let $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ be an acute or polyhedral convex cone. Then $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$.*

Proof. Since P is bounded, the efficient set of any deterministic MOLP is nonempty. Using Theorem 3.16 or 3.17, we obtain $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) \neq \emptyset$. □

While Theorem 3.17 and Corollary 3.19 address the special case that $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is polyhedral convex, in general, this cone is nonconvex since it is a union (rather than an intersection). Hence, we may not always be able to formulate an RC that is a deterministic MOLP as in Theorem 3.17. In particular, when the HRE set is disconnected, any RC would have at least one nonconvex objective (cf. Theorem 3.40, Ehrgott 2005). Despite these facts, as shown in Theorem 3.17, there exists a class of UMOLPs, those that have $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ being polyhedral convex, whose RC is a deterministic MOLP. Since MOLPs are readily solvable and their solution sets have desirable properties like connectedness, it is of interest to identify UMOLPs that have this characteristic. Consequently, recognizing the polyhedrality of $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ and computing its representation in order to obtain the cost matrix $\overline{\mathbf{C}}$ of RC (3.2) become important tasks. An algorithm to accomplish these two tasks is available in, e.g., Bemporad et al. (2001).

3.3. Acuteness Recognition and Discussion

Since the assumption of acuteness is key to several of the results we have already presented, it is important to examine this property in more detail. We first discuss the algebraic implication of the assumption that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute for at least one $\mathbf{u} \in U$. (Note that this discussion encompasses the deterministic context and $D_{\leqq}(\mathbf{C})$, as well as the situation that $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute since $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ being acute implies that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute for all $\mathbf{u} \in U$.) Since $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is closed, being acute is equivalent to being pointed by Proposition 2.1.4, Sawaragi et al. (1985). Hence, we implicitly assume that $\mathrm{rank}(\mathbf{C}(\mathbf{u})) = n$ by Theorem 3.1, Hunt et al. (2010). Since

$\text{rank}(\mathbf{C}(\mathbf{u})) \leq \min\{p, n\}$, we obtain that the number of criteria p is greater than or equal to the number of decision variables n. The consequence of this is that models that incorporate the numerous preferences of multiple decision makers explicitly through many criteria may be used.

Although algorithms are available to recognize polyhedrality, such methods have not been presented in the literature for recognizing the acuteness of a cone. It is worth noting that an acute cone need not be polyhedral (as it may not even be convex), and a polyhedral cone need not be acute. Hence, recognizing acuteness is a much different task than recognizing polyhedrality. We specifically examine an acuteness recognition method for the cones $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ for some $\mathbf{u} \in U$ (equivalently, $D_{\leqq}(\mathbf{C})$) and $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$.

Given the cone $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ for some $\mathbf{u} \in U$, we know that it may be expressed in both inequality form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}(\mathbf{u})\mathbf{x} \leqq \mathbf{0}\}$ (which is the form immediately available) and generator form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}(\mathbf{u})^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$, where $\mathbf{G}(\mathbf{u})^T$ is an $n \times r$ matrix whose columns are a finite set of generators of $D_{\leqq}(\mathbf{C}(\mathbf{u}))$. If $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is given in inequality form, then its polar is given in generator form as in Proposition 2.2(i). Similarly, if $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is given in generator form, then its polar is given in inequality form. Namely,

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{G}(\mathbf{u})^T\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}^+ = \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{G}(\mathbf{u})\mathbf{x} \leqq \mathbf{0}\}, \tag{3.4}$$

which follows from the statement on p. 90, Panik (1993). With this in mind, we have the following method for recognizing the acuteness of $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ for some $\mathbf{u} \in U$.

Theorem 3.20. *For some $\mathbf{u} \in U$, let $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ be given in generator form. Then $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute if and only if $-\mathbf{G}(\mathbf{u})\mathbf{x} < \mathbf{0}$ is consistent.*

Proof. Since $D_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$, we know that $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute if and only if $\text{int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) \neq \emptyset$ by Theorem 2.1(i), Yu (1974). As $\text{int}(\{\mathbf{x} \in \mathbb{R}^n : -\mathbf{G}(\mathbf{u})\mathbf{x} \leqq \mathbf{0}\}) = \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{G}(\mathbf{u})\mathbf{x} < \mathbf{0}\}$, the result follows from (3.4). □

More generally, we have a second recognition method given by the following theorem.

Theorem 3.21. *If $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = n$, then $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute.*

Proof. Let $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = n$. Hence, $\text{int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) = \text{rel}\,\text{int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u})))$ as on p. 44, Rockafellar (1970). Moreover, since $D^+_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ (as discussed earlier) and convex (see Proposition 2.1.5(i), Sawaragi et al. 1985), we obtain that $\text{rel}\,\text{int}(D^+_{\leqq}(\mathbf{C}(\mathbf{u}))) \neq \emptyset$ by Theorem 6.2, Rockafellar (1970). Thus, since $D_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \emptyset$, Theorem 2.1(i), Yu (1974), gives the result. □

Observe that Theorem 3.21 does not depend on the form, inequality or generator, of $D_{\leqq}(\mathbf{C}(\mathbf{u}))$, but instead relies on $\dim(D^{+}_{\leqq}(\mathbf{C}(\mathbf{u})))$. Even though we do not have a system to solve as in Theorem 3.20, we do have a condition to verify, namely that $\dim(D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}))) = n$. In particular, if $D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}))$ is in generator form (as it is when $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ is in inequality form), then $\dim(D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}))) = \operatorname{rank}(\mathbf{C}(\mathbf{u}))$ (see p. 86, Panik 1993). Otherwise, software such as SageMath's `polyhedron base class`, The Sage Developers (2017), can readily provide the dimension. We also note that Theorem 3.21 is applicable to any nonempty cone, while Theorem 3.20 is not. Using Theorems 3.20 and 3.21, we may similarly verify the acuteness of $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$.

Corollary 3.22. *Let $D_{\leqq}(\mathbf{C}(\mathbf{u}))$ be given in generator form for each $\mathbf{u} \in U$. Then $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute if and only if $-\mathbf{G}(\mathbf{u}^i)\mathbf{x} < \mathbf{0}$ is consistent for all $i = 1, \dots, s$.*

Proof. Follows from Theorem 3.20, Proposition 2.8(i), where $\mathbf{M}_k$ is replaced by $\mathbf{C}(\mathbf{u}^i)$, and the fact that $\operatorname{int}(\bigcap_{i=1}^{s} D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}^i))) = \bigcap_{i=1}^{s} \operatorname{int}(D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}^i)))$ (see p. 6, Steen & Seebach, Jr. 1970). $\square$

Likewise, we have the following extension of Theorem 3.21.

Proposition 3.23. *If $\dim(D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}), U)) = n$, then $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute.*

Proof. Follows similarly to the proof of Theorem 3.21. $\square$

It is important to note that when the proposed methods to verify acuteness are applied to $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$, they do not necessarily compute the cost matrix $\widetilde{\mathbf{C}}$ that appears in Theorem 3.16. Regardless, as a demonstration of both recognition methods, specifically Corollary 3.22 and Proposition 3.23, consider the following example.

Example 3.24. Consider UMOLP (3.1) with $U_1 = \{(1,1)\}$ and $U_2 = \{(1,1),(2,-1/9)\}$. We have two scenarios $\mathbf{u}^1 = (1,1,1,1)$ and $\mathbf{u}^2 = (1,1,2,-1/9)$. The generators of $D_{\leqq}(\mathbf{C}(\mathbf{u}^1))$ and $D_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ are $\mathbf{g}_1(\mathbf{u}^1) = \begin{bmatrix}-3 & -1\end{bmatrix}^T, \mathbf{g}_2(\mathbf{u}^1) = \begin{bmatrix}-9 & -1\end{bmatrix}^T, \mathbf{g}_1(\mathbf{u}^2) = \begin{bmatrix}3 & 1\end{bmatrix}^T$, and $\mathbf{g}_2(\mathbf{u}^2) = \begin{bmatrix}-1 & 2\end{bmatrix}^T$, respectively. Hence, the polars $D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}^1))$ and $D^{+}_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ are

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix}3 & 1\\ 9 & 1\end{bmatrix}\mathbf{x} \leqq \mathbf{0}\right\} \text{ and } \left\{\mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix}-3 & -1\\ 1 & -2\end{bmatrix}\mathbf{x} \leqq \mathbf{0}\right\},$$

respectively, by (3.4). Applying Corollary 3.22, $D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cup D_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ is acute if and only if the system given by

$$\begin{aligned} 3x_1 + x_2 &< 0 \\ 9x_1 + x_2 &< 0 \\ -3x_1 - x_2 &< 0 \\ x_1 - 2x_2 &< 0 \end{aligned} \tag{3.5}$$

is consistent. It is clear that (3.5) is inconsistent as the first and third inequalities are inconsistent. Thus, as confirmed in Figure 3a, $D_{\leqq}(\mathbf{C}(\mathbf{u}^1)) \cup D_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ is not acute.

Moreover, note that $D^+_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is the ray in the second quadrant emanating from the origin with slope -3 (cf. Figure 3b). Hence, $\dim(D^+_{\leqq}(\mathbf{C}(\mathbf{u}), U)) = 1 \neq n$. With Proposition 3.23 in mind, this means that we should not expect $D_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ to be acute.

4. Conclusion

In this paper, we have presented the first in-depth analysis of HRE solutions to objective-wise UMOLPs under finite sets of scenarios. The assumed objective-wise uncertainty has three main benefits including that it permits (1) the model to incorporate the practical reality that conflicting criteria are unlikely to depend on the same uncertainty, (2) interval multiobjective linear programming to be considered as special case, and (3) the application of an existing polytopal uncertainty set reduction, which consequently motivates the use of finite sets of scenarios.

We first develop a variety of properties of the HRE set including clear extensions of known results regarding the efficient set of a deterministic MOLP. More importantly, under a condition of acuteness, we provide a characterization of the HRE set by means of the normal cone and the strict polar of the closed cone of improving directions associated with every instance of the UMOLP. The acuteness of the closed cone of improving directions also leads to a lower bound set on the HRE set and guarantees that the HRE set is nonempty. Furthermore, the polyhedrality of the latter cone leads to an MOLP that is an RC of the UMOLP. The polyhedrality of the cone may be verified and its algebraic representation computed by an existing algorithm that immediately leads to a closed form representation of the previously mentioned RC. On the other hand, acuteness of the cone may be checked by either of two proposed methods, solving a system of linear inequalities or computing the dimension of the cone, both of which are easily performed using readily available software.

Our work immediately opens up several avenues for continued research. In particular, methods to compute HRE solutions, as well as the entire HRE set, remain to be developed in the absence of the above polyhedrality condition. Additionally, an algorithm to determine the cost matrix of the MOLP whose efficient set is a lower bound set on the HRE set is needed. Finally, it is also desirable to relax the acuteness condition in order to address a more general class of UMOLPs.

Acknowledgement

Margaret M. Wiecek gratefully acknowledges support by the United States Office of Naval Research through grant number N00014-16-1-2725.

References

Bemporad, A., Fukuda, K., & Torrisi, F. (2001). Convexity recognition of the union of polyhedra. *Computational Geometry*, *18*, 141–154.

Ben-Tal, A., & Nemirovski, A. (1998). Robust convex optimization. *Mathematics of Operations Research*, *23*, 769–805.

Benson, H. (1985). Multiple objective linear programming with parametric criteria coefficients. *Management Science*, *31*, 461–474.

Bitran, G. (1980). Linear multiple objective problems with interval coefficients. *Management Science*, *26*, 694–706.

Bronson, R. (1989). *Schaum's Outline of Theory and Problems of Matrix Operations*. McGraw-Hill.

Carothers, N. (2000). *Real Analysis*. Cambridge University Press.

Dattorro, J. (2015). *Convex Optimizatoin & Euclidean Distance Geometry*. Palo Alto, CA: Meboo Publishing USA.

Ehrgott, M. (2005). *Multicriteria Optimization*. (2nd ed.). Springer.

Ehrgott, M., Ide, J., & Schöbel, A. (2014). Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, *239*, 17–31.

Engau, A., & Wiecek, M. (2008). Interactive coordination of objective decompositions in multiobjective programming. *Management Science*, *54*, 1350–1363.

Goberna, M., Jeyakumar, V., Li, G., & Vicente-Pérez, J. (2015). Robust solutions to multi-objective linear programs with uncertain data. *European Journal of Operational Research*, *242*, 730–743.

Greer, R. (1984). *Trees and Hills: Methodology for Maximizing Functions of Systems of Linear Relations* volume 22 of *Annals of Discrete Mathematics*. Amsterdam, The Netherlands: Elsevier Science Publishers.

Hladík, M. (2010). On necessarily efficient solutions in interval multiobjective linear programming. In *Proceedings of the 25th Mini-EURO Conference on Uncertainty and Robustness in Planning and Decision Making* (pp. 1–10).

Hladík, M. (2012). Complexity of necessary efficiency in interval linear programming and multiobjective linear programming. *Optimization Letters*, *6*, 893–899.

Hunt, B., Wiecek, M., & Hughes, C. (2010). Relative importance of criteria in multiobjective programming: A cone-based approach. *European Journal of Operational Research*, *207*, 936–945.

Ida, M. (1996). Generation of efficient solutions for multiobjective linear programming with interval coefficients. In *SICE'96. Proceedings of the 35th SICE Annual Conference. International Session Papers* (pp. 1041–1044). IEEE.

Ide, J., & Schöbel, A. (2015). Robustness for uncertain multi-objective optimization: A survey and analysis of different concepts. *OR Spectrum*, *38*, 235–271.

Inuiguchi, M., & Kume, Y. (1991). Dominance relations as bases for constructing solution concepts in linear programming with multiple interval objective functions. *Bulletin of the University of Osaka Prefecture*, *40*, 275–292.

Inuiguchi, M., & Kume, Y. (1992). Efficient solutions versus nondominated solutions in linear programming with multiple interval objective functions. *Bulletin of the University of Osaka Prefecture*, *40*, 293–298.

Inuiguchi, M., & Sakawa, M. (1996). Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test. *Fuzzy Sets and Systems*, *78*, 231–241.

Kuhn, K., Raith, A., Schmidt, M., & Schöbel, A. (2016). Bi-objective robust optimisation. *European Journal of Operational Research*, *252*, 418–431.

Luc, D. (2016). *Multiobjective Linear Programming: An Introduction*. Springer International Publishing.

Mangasarian, O. (1969). *Nonlinear Programming*. Systems Science. McGraw-Hill.

Matheron, G. (1975). *Random Sets and Integral Geometry*. John Wiley & Sons.

Oliveira, C., & Antunes, C. (2007). Multiple objective linear programming models with interval coefficients–an illustrated overview. *European Journal of Operational Research*, *181*, 1434–1463.

Panik, M. (1993). *Fundamentals of Convex Analysis*. Kluwer Academic Publishers.

Rockafellar, R. (1970). *Convex Analysis*. Princeton University Press.

Sawaragi, Y., Nakayama, H., & Tanino, T. (1985). *Theory of Multiobjective Optimization* volume 176 of *Mathematics in Science and Engineering*. Academic Press.

Steen, L., & Seebach, Jr., J. (1970). *Counterexamples in Topology*. USA: Holt, Rinehart and Winston, Inc.

The Sage Developers (2017). *SageMath, the Sage Mathematics Software System*. URL: http://www.sagemath.org.

Thoai, N. (2012). Criteria and dimension reduction of linear multiple criteria optimization problems. *Journal of Global Optimization*, *52*, 499–508.

Wiecek, M., & Dranichak, G. (2016). Robust multiobjective optimization for decision making under uncertainty and conflict. In A. Gupta, A. Capponi, & J. Smith (Eds.), *Optimization Challenges in Complex, Networked, and Risky Systems* INFORMS TutORials in Operations Research chapter 4. (pp. 84–114). The Sheridan Press.

Yu, P. (1974). Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. *Journal of Optimization Theory and Applications*, *14*, 319–377.