MATH 231A
Solutions to PS 15

<u>7.8/#2</u> This matrix has just one eigenvalue $\lambda = 0$ of multiplicity two, and this eigenvalue is deficient because it yields just one linearly independent eigenvector $\boldsymbol{\xi} = (1, 2)$. Thus, one solution of the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}^{(1)}(t) = e^{0\cdot t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We need to find a second solution of the form $\mathbf{x}^{(2)}(t) = e^{0\cdot t}(\boldsymbol{\eta}+t\boldsymbol{\xi}) = \boldsymbol{\eta}+t\boldsymbol{\xi}$, where $(A-0I)\boldsymbol{\eta} = A\boldsymbol{\eta} = \boldsymbol{\xi}$. That is,

$$\left.\begin{aligned} 4\eta_1 - 2\eta_2 &= 1 \\ 8\eta_1 - 4\eta_2 &= 2 \end{aligned}\right\} \quad \Rightarrow \quad 4\eta_1 - 2\eta_2 = 1 \qquad \text{(same equation)}$$

$$\Rightarrow \quad \eta_2 = 2\eta_1 - \frac{1}{2}.$$

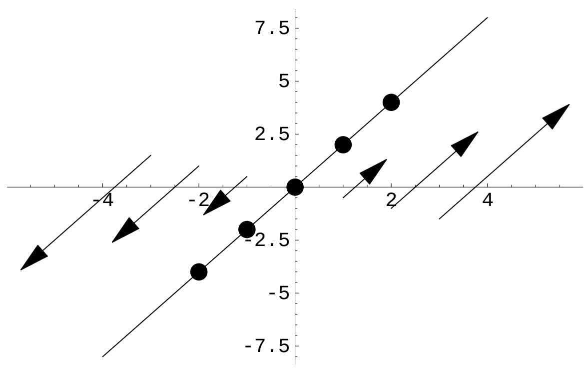
Taking $\eta_1 = \alpha$ (free), we have infinitely many such vectors $\boldsymbol{\eta}$, all of the form

$$\boldsymbol{\eta} = \left(\alpha, 2\alpha - \frac{1}{2}\right) = \alpha(1, 2) + \left(0, -\frac{1}{2}\right).$$

We only need one such $\boldsymbol{\eta}$, which will be result of choosing a particular value for β. The easiest thing is to take $\beta = 0$, which I will do in all future problems. To illustrate that any β is allowable, I will take $\beta = 1$ in this case, yielding $\boldsymbol{\eta} = (1, 3/2)$. Thus, the general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 3/2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right].$$

Any particular solution with $c_2 = 0$, will remain (constantly) $c_1(1, 2)$ — that is, if a solution begins along the line through the origin in the direction of $(1, 2)$, then it forever stays right where it started. (This line consists of only equilibrium points.) Solutions with $c_2 \neq 0$ (i.e., which do not start on the above-mentioned line) do vary with time. However, all vary in precisely the same fashion moving in the direction of $(1, 2)$ if $c_2 > 0$ and in the opposite direction if $c_2 < 0$. Those with a larger (in magnitude) choice of c_2 will move faster than those with a smaller c_2. Thus, we get solution trajectories that appear as below (longer arrows correspond to faster-moving trajectories).



<u>7.8/#3</u> This matrix has the repeated eigenvalue $\lambda = -1$, also deficient in that it yields just one linearly independent eigenvector $\boldsymbol{\xi} = (2, 1)$. The system of equations for $\boldsymbol{\eta}$ (what the book calls a *generalized eigenvector*) $(A + I)\boldsymbol{\eta} = \boldsymbol{\xi}$ can be written out as

$$\left.\begin{aligned} -\frac{1}{2}\eta_1 + \eta_2 &= 2 \\ -\frac{1}{4}\eta_1 + \frac{1}{2}\eta_2 &= 1 \end{aligned}\right\} \quad \Rightarrow \quad -\frac{1}{2}\eta_1 + \eta_2 = 2 \qquad \text{(same equation)}$$

$$\Rightarrow \quad \eta_1 = 2\eta_2 - 4.$$

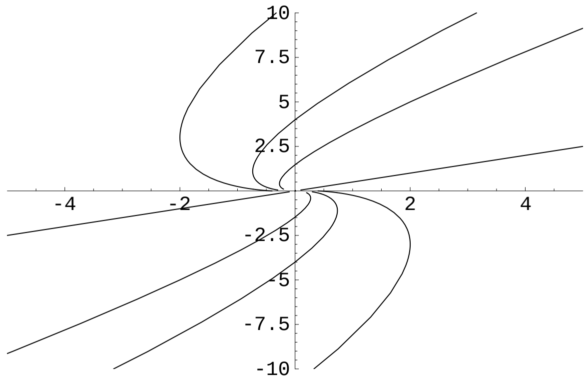
Taking η_2 to be free, and setting it equal to zero, we get $\boldsymbol{\eta} = (-4, 0)$. Thus, the general solution is

$$\mathbf{x}(t) \;=\; c_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t}\left[\begin{pmatrix} 0 \\ -4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right].$$

From this general solution, we see that all solutions are attracted to the origin as $t \to \infty$. In particular, those solutions that begin on the line through the origin in the direction $(2, 1)$ (i.e., those solutions for which $c_2 = 0$), go straight along that line towards the origin (though they will take forever to get there). The solutions that start off this line (those which correspond to a nonzero c_2) have more interesting behavior. As $t \to \infty$, both parts of $\mathbf{x}^{(2)}$ become negligible. However, the term

$$c_2 e^{-t}\begin{pmatrix} 0 \\ -4 \end{pmatrix} \qquad \text{dies off faster than} \qquad c_2 t e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus, such solutions asymptotically approach the previously-mentioned line as $t \to \infty$. If we follow such trajectories backward in time rather than forward (i.e., let $t \to -\infty$), then both terms of $\mathbf{x}^{(2)}$ grow exponentially. Nevertheless, $c_2 t^e - t(2, 1)$ still dominates the other term, and though such trajectories do not approach our previous line asymptotically as $t \to -\infty$, they do become more and more parallel to it.



7.8/#6 This matrix has two eigenvalues: $\lambda = 2$ with its associated eigenvector $(1, 1, 1)$, and $\lambda = -1$ (multiplicity 2) with two linearly independent eigenvectors $(-1, 0, 1)$ and $(-1, 1, 0)$ (so $\lambda = -1$ is *not* a deficient eigenvalue). Thus, the general solution is

$$\mathbf{x}(t) \;=\; c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

7.8/#7 This matrix has the repeated eigenvalue $\lambda = -3$, but is deficient, having just the one (representative) linearly independent eigenvector $(1, 1)$. There is, then, the need to find a solution of the form $e^{-3t}(\boldsymbol{\eta} + t(1, 1))$. We seek $\boldsymbol{\eta}$ by solving $(A + 3I)\boldsymbol{\eta} = (1, 1)$:

$$\left.\begin{aligned} 4\eta_1 - 4\eta_2 &= 1 \\ 4\eta_1 - 4\eta_2 &= 1 \end{aligned}\right\} \qquad \Rightarrow \qquad \eta_1 \;=\; \eta 2 + \frac{1}{4}.$$

Taking η_2 (free) to be zero, we have $\boldsymbol{\eta} = (1/4, 0)$. Thus, the general solution is

$$\begin{aligned} \mathbf{x}(t) \;&=\; c_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\left[\begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \\ &=\; e^{-3t}\begin{pmatrix} c_1 + c_2(t + 1/4) \\ c_1 + c_2 t \end{pmatrix}. \end{aligned}$$

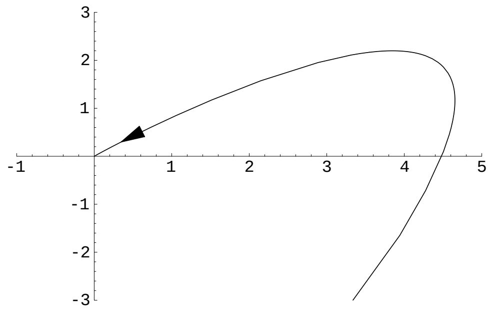
At time $t = 0$, we need this vector to be $(3, 2)$, and so we solve

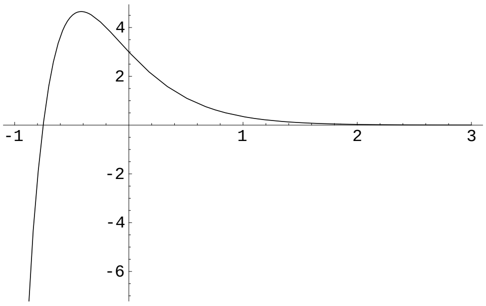
$$\begin{pmatrix} c_1 + (1/4)c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

getting $c_1 = 2$ and $c_2 = 4$. Thus, the IVP has solution

$$\mathbf{x}(t) \;=\; e^{-3t}\begin{pmatrix} 2+4(t+1/4) \\ 2+4t \end{pmatrix} \;=\; e^{-3t}\begin{pmatrix} 3+4t \\ 2+4t \end{pmatrix}.$$

Mathematica's **ParametricPlot**[ ] command may be used to draw the solution's trajectory in the phase plane (x_1x_2-plane); see left below. The graph on the right is that of $x_1(t) = e^{-3t}(3+4t)$.





<u>7.8/#17(d)</u> The third solution's form has already been suggested in the problem:

$$\mathbf{x}^{(3)}(t) \;=\; e^{2t}\left(\boldsymbol{\zeta} + t\boldsymbol{\eta} + \frac{t^2}{2}\boldsymbol{\xi}\right).$$

We will not assume, at the outset, anything in particular about $\boldsymbol{\xi}$ or $\boldsymbol{\eta}$ (that is, we are not assuming that $\boldsymbol{\xi}$ solves $(A+2I)\boldsymbol{\xi} = \mathbf{0}$ nor that $\boldsymbol{\eta}$ solves $(A+2I)\boldsymbol{\eta} = \boldsymbol{\xi}$). What we do assume is that $x^{(3)}$ solves $\mathbf{x}' = A\mathbf{x}$. This means that, $\dfrac{d}{dt}\mathbf{x}^{(3)} = A\mathbf{x}^{(3)}$, or

$$e^{2t}\left(2\boldsymbol{\zeta} + 2t\boldsymbol{\eta} + \boldsymbol{\eta} + t^2\boldsymbol{\xi} + t\boldsymbol{\xi}\right) \;=\; e^{2t}\left(A\boldsymbol{\zeta} + tA\boldsymbol{\eta} + \frac{t^2}{2}A\boldsymbol{\xi}\right).$$

Dividing through by e^{2t} and grouping some terms, we have

$$(A-2I)\boldsymbol{\zeta} + t(A-2I)\boldsymbol{\eta} + \frac{t^2}{2}(A-2I)\boldsymbol{\xi} \;=\; \boldsymbol{\eta} + t\boldsymbol{\xi},$$

an equation which must hold for all t. Because of this, the coefficients of the t^2 terms must match, as well as those of the t and constant terms. That is,

$$(A-2I)\boldsymbol{\xi} \;=\; \mathbf{0}, \qquad (A-2I)\boldsymbol{\eta} \;=\; \boldsymbol{\xi}, \qquad \text{and} \qquad (A-2I)\boldsymbol{\zeta} \;=\; \boldsymbol{\eta}.$$

So, though we did not presume it to be the case, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ do solve the usual eigenvector and generalized eigenvector equations, while $\boldsymbol{\zeta}$ solves a similar equation as the one that $\boldsymbol{\eta}$ solves, only with right-hand side η instead of $\boldsymbol{\xi}$.

<u>*</u> The given matrix has the triple eigenvalue $\lambda = -5$, whose only linearly independent eigenvector is $(-3,1,2)$. Thus we must solve $(A+5I)\boldsymbol{\eta} = (-3,1,2)$ for $\boldsymbol{\eta}$, and then solve $(A+5I)\boldsymbol{\zeta} = \boldsymbol{\eta}$ for $\boldsymbol{\zeta}$. The first of these, using Gaussian elimination, is

$$\left(\begin{array}{ccc|c} -2 & 4 & -5 & -3 \\ -1 & -3 & 0 & 1 \\ 3 & -1 & 5 & 2 \end{array}\right) \qquad \overset{\mathbf{r}_1 \leftrightarrow \mathbf{r}_2}{\sim} \qquad \left(\begin{array}{ccc|c} -1 & -3 & 0 & 1 \\ -2 & 4 & -5 & -3 \\ 3 & -1 & 5 & 2 \end{array}\right)$$

$$\begin{array}{c} (-\mathbf{r}_1) \to \mathbf{r}_1 \\ -2\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}2 \\ \sim \\ 3\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 10 & -5 & -5 \\ 0 & -10 & 5 & 5 \end{array} \right)$$

$$\begin{array}{c} \mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}3 \\ \sim \end{array} \quad \left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 10 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Taking η_3 (free) to be zero, the second row says

$$10\eta_2 - 5(0) \;=\; -5 \qquad \Rightarrow \qquad \eta_2 \;=\; -\frac{1}{2},$$

and thus the first row says

$$\eta_1 - 3\left(\frac{1}{2}\right) \;=\; -1 \qquad \Rightarrow \qquad \eta_1 \;=\; \frac{1}{2}.$$

So, $\boldsymbol{\eta} = (1/2, -1/2, 0)$.

Now, we solve for $\boldsymbol{\zeta}$:

$$\left(\begin{array}{ccc|c} -2 & 4 & -5 & 1/2 \\ -1 & -3 & 0 & -1/2 \\ 3 & -1 & 5 & 0 \end{array} \right) \quad \begin{array}{c} \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \\ \sim \end{array} \quad \left(\begin{array}{ccc|c} -1 & -3 & 0 & -1/2 \\ -2 & 4 & -5 & 1/2 \\ 3 & -1 & 5 & 0 \end{array} \right)$$

$$\begin{array}{c} (-\mathbf{r}_1) \to \mathbf{r}_1 \\ -2\mathbf{r}_1 + \mathbf{r}_2 \to \mathbf{r}2 \\ \sim \\ 3\mathbf{r}_1 + \mathbf{r}_3 \to \mathbf{r}3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1/2 \\ 0 & 10 & -5 & 3/2 \\ 0 & -10 & 5 & -3/2 \end{array} \right)$$

$$\begin{array}{c} \mathbf{r}_2 + \mathbf{r}_3 \to \mathbf{r}3 \\ \sim \end{array} \quad \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1/2 \\ 0 & 10 & -5 & 3/2 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Once again, ζ_3 is free, and we take it to be zero. Thus,

$$10\zeta_2 - 5(0) \;=\; \frac{3}{2} \qquad \Rightarrow \qquad \zeta_2 \;=\; \frac{3}{20},$$

and

$$\zeta_1 + 3\left(\frac{3}{20}\right) \;=\; \frac{1}{2} \qquad \Rightarrow \qquad \zeta_1 \;=\; \frac{1}{20}.$$

So, $\boldsymbol{\zeta} = (1/20, 3/20, 0)$, and the general solution is

$$\begin{aligned} \mathbf{x}(t) \;=\; e^{-5t} \Bigg\{ & c_1 \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \right] \\ & + c_3 \left[\begin{pmatrix} 1/20 \\ 3/20 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \right] \Bigg\}. \end{aligned}$$