E20-151

THE VLASOV EQUATIONS

N66 24978

N66 24978 (ACCESSION NUMBER)	(THRU)
89 (PAGES)	1 (CODE)
CR-74673 (NASA CR OR TMX OR AD NUMBER)	25 (CATEGORY)

FACILITY FORM 602

RESEARCH & ANALYSIS SECTION TECH. MEMO. NO. 137

By

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PREPARED UNDER CONTRACT NAS8-11440

NOVEMBER 1965

For:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ACKNOWLEDGEMENT

The author would like to express his sincere appreciation to Dr. K. S. Ram who patiently read the manuscript, and to Mr. H. R. Rosner for his constant encouragement.

Many thanks are also due to Mr. George Fichtl with whom he had many discussions. He would also like to acknowledge the continued interest and financial support of the NASA Marshall Space Flight Center at Huntsville, Alabama.

ABSTRACT

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As a preliminary step in investigating the dynamics of a collisionless plasma, a comprehensive literature survey is conducted. Emphasis is placed on those aspects of the recent theoretical development which may shed light on a better approximation for the solution of the Vlasov equations.

All relevant equations of classical mechanics, classical electrodynamics, and hydrodynamics are either displayed in a suggestive manner or derived from fundamental principles and incorporated in the text of this report.

A brief but self-contained review of kinetic theory is given, and the microscopic hydrodynamical equations are derived on the basis of the Boltzmann equation for a dilute neutral gas based on a hard sphere model.

Classical statistical mechanics is cast in a form which is particularly suited for its application to plasma dynamics. Liouville's theorem is applied to a relativistic electron gas with a positive ion background. The Vlasov-Boltzmann Equation is obtained from Liouville's theorem on the basis of the self-consistent field formalism of Rostoker and Rosenbluth.

The Born-Bogoliubov-Green-Kirkwood-Yvon Hierarchy is derived from the relativistic Liouville's equation. Corrections to the Vlasov-Boltzmann Equation in terms of the multiparticle correlation function technique are also discussed.

Some recent results on non-linear oscillations, plasma radiation, and conductivity will be surveyed in the next report. A generalization of integral-of-the-motion method to include relativity is also planned.

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CHAPTER ONE

INTRODUCTION

I-1 Classical Mechanics in Canonical Form

A. Lagrangian Formulation

We consider first a dynamical system consisting of n particles having coordinates q_i $(i = 1,2,...n)$ and described by the Lagrangian

$$\mathcal{L}(q_i, \dot{q}_i, t), \quad i = 1, 2, \cdots n.$$

The Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad i = 1, 2, \cdots n \qquad \text{A-1}$$

are obtained from Hamilton's principle

$$\delta \int \mathcal{L}(q_i, \dot{q}_i, t)\, dt = 0 \qquad \text{A-2}$$

with

$$\delta q_i \,(\text{end points}) = 0$$

B. Hamilton's Formulation

The Hamiltonian of a dynamical system of particles is defined in terms of the generalized coordinates and momenta, and is related to the Lagrangian by the Legendre transformation

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} \tag{B-1}$$

where

$$\dot{q}_i = \frac{dq_i}{dt},$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad i = 1, 2, \cdots n$$

Then

$$\mathcal{H} = \mathcal{H}(p_i, q_i, t),$$

and Hamilton's equations of motion are

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \tag{B-2}$$

Consider next an arbitrary function $F(p_i q_i t)$ (with $i = 1, 2, \ldots n$) and its time derivative

$$\frac{dF}{dt} = \sum_i \left[\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right] + \frac{\partial F}{\partial t}$$

$$= \sum_i \left[\frac{\partial F}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right] + \frac{\partial F}{\partial t} .$$

Introducing the classical Poisson bracket

$$[F, \mathcal{H}] = \sum_i \left[\frac{\partial F}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right],$$

we then have

$$\frac{dF}{dt} = [F, \mathcal{H}] + \frac{\partial F}{\partial t} \qquad B\text{-}3$$

I-2 Classical Electrodynamics

A. The Maxwell Equations

For the sake of completeness and future reference, let us review here very briefly some of the basic features of Maxwell's equations. In Gaussian units these are

$$\begin{aligned} &\nabla \cdot \vec{B} = 0 , \quad \nabla \cdot \vec{D} = 4\pi\rho , \\ &\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} , \quad \vec{B} = \mu \vec{H} , \quad \vec{D} = K\vec{E} , \qquad A-1 \\ &\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \end{aligned}$$

where $\vec{B}$ is the magnetic field,† $\vec{E}$ the electric field, $\vec{J}$ is the current density, and ρ is the charge density. It follows from the second and the last of the Maxwell Equations that

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 ,$$

† For applications to plasma physics $\mu = 1$, $K = 1$, and $B = H$, $D = E$ the free space approximation is sufficient, for this point, see, for example, Reference 17.

which is the equation of continuity. The electromagnetic fields can be conveniently expressed in terms of the scalar and vector potentials, φ and $\vec{A}$ respectively, and these are defined by the equations

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \because \ \nabla \cdot \vec{B} = 0$$
$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \because \ \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \qquad A-2$$

However, these equations do not define φ and $\vec{A}$ uniquely. For instance, the gauge transformation

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$
$$\varphi \longrightarrow \varphi' = \varphi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \quad \Lambda = \Lambda(\vec{r}, t) \qquad A-3$$

where Λ is an arbitrary scalar function of space and time, leaves $\vec{E}$ and $\vec{B}$ invariant.

B. The Lorentz Gauge and the Coulomb Gauge

The inhomogeneous Maxwell equations can be written in terms of the scalar and vector potentials as

$$\nabla^2\varphi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \vec{A} = -4\pi\rho$$
$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t}\right) = -\frac{4\pi}{c}\vec{J} \qquad B-1$$

In view of the freedom implied by the gauge transformation we can choose

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \quad (\mathcal{Lorentz\ Gauge})$$

so as to uncouple the pair of equations B-1, thus:

$$\begin{aligned} \nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} &= -4\pi\rho \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{J} \end{aligned} \qquad B-2$$

with solutions

$$\begin{aligned} \varphi(\vec{r},t) &= \int d^3r' \frac{\rho[\vec{r}',t^*]}{|\vec{r}-\vec{r}'|}{}^{\dagger}, \\ \vec{A}(\vec{r},t) &= \frac{1}{c}\int d^3r' \frac{\vec{J}[\vec{r}',t^*]}{|\vec{r}-\vec{r}'|} \end{aligned} \qquad B-3$$

where $t^* = t - \frac{|\vec{r}-\vec{r}'|}{c}$ is the retarded time if $\vec{J}(\vec{r},t)$ and $\rho(\vec{r},t)$ are given. The Lorentz gauge (or restricted gauge) transformation is thus given by

$$\begin{aligned} &\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \\ &\varphi \longrightarrow \varphi' = \varphi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \\ &\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0, \quad \nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \end{aligned} \qquad B-4$$

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† For a derivation see, for example, J. D. Jackson, Classical Electrodynamics, pages 183-186.

Another useful gauge is the transverse or Coulomb gauge, defined by

$$\nabla \cdot \vec{A} = 0$$

The potentials then satisfy the following equations:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}_t ,$$
$$\nabla^2 \varphi = -4\pi\rho, \quad \varphi(\vec{r},t) = \int d^3r' \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} \qquad \text{B-5}$$

where the transverse current $\vec{J}_t$ is given by

$$\vec{J}_t = \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int d^3r' \frac{\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

with

$$\vec{J} = \vec{J}_\ell + \vec{J}_t$$

and

$$\vec{J}_\ell = -\frac{1}{4\pi}\vec{\nabla}\int \frac{\nabla'\cdot\vec{J}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\, d^3r' \qquad \text{B-6}$$

with

$$\vec{\nabla}\cdot\vec{J}_t = 0, \quad \vec{\nabla}\times\vec{J}_\ell = 0$$

If

$$\rho = 0, \ \vec{J} = 0, \ \text{then } \varphi = 0, \ \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \ \vec{B} = \vec{\nabla}\times\vec{A}.$$

C. The Lorentz Force Equation and Electromagnetic Stress Tensor

The electromagnetic force on a charged particle is given by the Lorentz force

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \qquad \text{C-1}$$

From Newton's second law we can write the rate of change of the particle's momentum as

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

For a system of charged particles, we have

$$\frac{d}{dt}\vec{P}_{mech.} = \int d^3r \left[\rho \vec{E} + \frac{1}{c}\vec{J} \times \vec{B}\right] \qquad \text{C-2}$$

We have converted the sum over particles to an integral over charge and current densities for convenience. The particle nature can be recovered at any stage by making use of delta functions. It is shown in reference (9) that the Lorentz force equation leads to the conservation of linear momentum, thus:

$$\frac{d}{dt}\left[\vec{P}_{mech.} + \vec{P}_{field}\right] = \oint_S \hat{n} \cdot \overleftrightarrow{T}\, dS \qquad \text{C-3}$$

where

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left[\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}\overleftrightarrow{I}(E^2+B^2)\right],$$

$$\overleftrightarrow{I} = \hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2 + \hat{e}_3\hat{e}_3$$

$$\overleftrightarrow{T} = \sum_{ij} \hat{e}_i T_{ij} \hat{e}_j = \hat{e}_i T_{ij} \hat{e}_j \ (\textit{Einstein Convention})$$

$$T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E^2+B^2)\right],$$

$$\vec{P}_{field} = \frac{1}{4\pi c}\int (\vec{E} \times \vec{B})\, d^3r$$

I-3 The Hydrodynamical Equations

The classical hydrodynamical equations for a neutral non-conducting fluid are

$$\nabla\cdot(\rho\vec{V}) + \frac{\partial\rho}{\partial t} = 0 \quad \text{(Equations of Continuity)} \qquad 1$$

where $\rho = \frac{M}{V} = \frac{Nm}{V} = nm$ = density

N = total number of particles in a volume V

m = mass of each particle,

and

$$\rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}\cdot\overleftrightarrow{T} + \vec{F} \qquad 2$$

(Stokes-Navier Equation)

where

$\vec{V}$ = macroscopic fluid velocity

$\overleftrightarrow{T} = \hat{e}_i T_{ij}\hat{e}_j$ = stress tensor

$$T_{ij} = P\,\delta_{ij} + \lambda\nabla\cdot\vec{V}\;\delta_{ij} + \mu\left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\right)$$

Consequently

$$\rho\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = \vec{F} - \vec{\nabla}P + (\lambda+\mu)\vec{\nabla}(\nabla\cdot\vec{V}) + \mu\nabla^2\vec{V}^{\dagger}$$

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† An elementary demonstration is given in Reference 14.

If $\overleftrightarrow{T}$ is diagonal and $\nabla \cdot \vec{V} = 0$, this becomes Euler's Equation:

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \vec{F} \qquad 3$$

We also have the heat flow equation,

$$n\left(\frac{\partial \epsilon}{\partial t} + (\vec{V}\cdot\vec{\nabla})\epsilon\right) = -\overleftrightarrow{T} : \overleftrightarrow{D} - \vec{\nabla}\cdot\overleftrightarrow{Q} \;\dagger$$

or

$$n\frac{\partial \epsilon}{\partial t} + nV_i\frac{\partial \epsilon}{\partial x_i} = -T_{ij}D_{ij} - \frac{\partial Q_i}{\partial x_i} \qquad 4$$

where :

$\overleftrightarrow{Q}$ = heat flow tensor

$\overleftrightarrow{T}$ = stress tensor

$\overleftrightarrow{D}$ = deformation tensor

ϵ = thermal energy density

I-4. Boltzmann Equation for a Dilute Neutral Gas

A. Boltzmann Equation

We consider a six-dimensional space D_6 whose coordinates are the position and velocity of any molecule in a gas. The coordinate vector $\vec{\zeta}$ of a point P in this space is given by

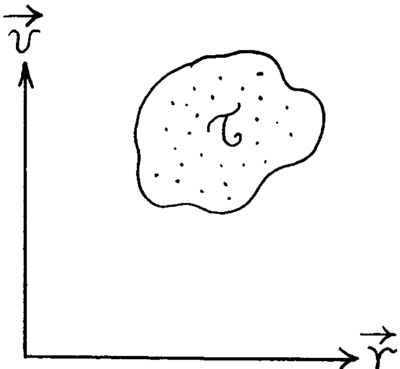
$$\vec{\zeta} = \sum_{i=1}^{6} (\vec{\zeta}\cdot\hat{\zeta}_i)\,\hat{\zeta}_i$$

† The hydrodynamical equations will be derived from an atomistic point of view in Section I-5B.

where

$$\sum_{i=1}^{3} \zeta_i \hat{\zeta}_i = \vec{r},$$

$$\sum_{i=4}^{6} \zeta_i \hat{\zeta}_i = \vec{v}, \quad \zeta_i = \vec{\zeta} \cdot \hat{\zeta}_i \, .$$



μ-Space

The point P determines the state of a single molecule by specifying its position and velocity.

The number of molecules which are in a volume τ in this space are given by the distribution function

$$F(\zeta_i \, t) \, d^6 \zeta_i = F(\vec{r}, \vec{v}, t) \, d^3 r \, d^3 v \qquad 1$$

The state of the gas can thus be described in terms of the motion of coordinate points in this space (D_6). The velocity of a point in D_6 is

$$\frac{d\vec{\zeta}}{dt} = \sum_{i=1}^{6} \dot{\zeta}_i \hat{\zeta}_i = \sum_{i=1}^{3} (\dot{\vec{r}} \cdot \hat{\zeta}_i) \hat{\zeta}_i + \sum_{i=4}^{6} (\dot{\vec{v}} \cdot \hat{\zeta}_i) \hat{\zeta}_i \qquad 2$$

Thus, each point in D_6 (corresponding to a molecule in physical space) has a velocity in each of its six coordinates. In the approximation of a short range molecular force, (i.e. the effective length of molecular interactions is small compared to the average interparticle distance), only close particles can collide, and when they do collide, particles which are in the space part of τ but not in the velocity part of τ can collide with particles in τ. On the other hand, particles in the velocity part of $\mathcal{T}$ but not in the space part of $\mathcal{T}$ cannot collide with particles in $\mathcal{T}$ due to our assertation of close collision resulting from the short range force approximation. It is clear that the number of particles in $\mathcal{T}$ may be increased or decreased because of collision.

The total time rate of change is given by

$$\frac{d}{dt}\int F d^6\zeta = -\sum_{i=1}^{6}\int F\dot{\zeta}_i d^6S_i + \int\left(\frac{\partial F}{\partial t}\right)_{Collision} d^6\zeta_i$$

$$= -\int\sum_{i=1}^{6}\frac{\partial}{\partial\zeta_i}(F\dot{\zeta}_i)\, d^6\zeta_i + \int\left(\frac{\partial F}{\partial t}\right)_{Collision} d^6\zeta_i$$

or $$\frac{dF}{dt} = -\sum_{i=1}^{6}\frac{\partial}{\partial\zeta_i}(F\dot{\zeta}_i) + \left(\frac{\partial F}{\partial t}\right)_{Collision}$$

$$\frac{dF}{dt} + \sum_{i=1}^{6}\frac{\partial}{\partial\zeta_i}(F\dot{\zeta}_i) = \left(\frac{\partial F}{\partial t}\right)_{Collision}$$

i.e. $$\frac{dF}{dt} + \nabla\cdot(F\vec{v}) + \nabla_{\vec{v}}\cdot(F\vec{a}) = \left(\frac{\partial F}{\partial t}\right)_{Collision}$$

B. Collision Term

We next evaluate the collision term explicitly. The number of collisions that occur between a particle of velocity $\vec{v}$ and one of velocity $\vec{v}_1$, leading to velocities $\vec{v}'$ and $\vec{v}_1'$, is given by

$$F(\vec{v})F(\vec{v}_1)\, d^3v\, d^3v_1\, d\sigma\, |\vec{v}-\vec{v}_1|$$

where

$$d\sigma = I(\Theta)\, d\Omega \,, \quad I(\Theta) = \left| \frac{b\,db}{\sin\Theta\, d\Theta} \right|$$

with the following meanings for the notation used:

$d\sigma$ = differential collision cross section

b = impact parameter

Θ = angle of scattering

$d\Omega$ = solid angle

Similarly the number of collisions starting with $\vec{v}\,'$ and $\vec{v}_1\,'$, and leading to $\vec{v}$ and $\vec{v}_1$, is

$$F(\vec{v}\,')F(\vec{v}_1\,')\,d^3v'\,d^3v_1'\,d\sigma\,|\vec{v}\,'-\vec{v}_1\,'|$$

Assuming elastically interacting particles of identical mass, then

$$d^3v\,d^3v_1\,d\sigma\,|\vec{v}-\vec{v}_1| = d^3v'\,d^3v_1'\,d\sigma\,|\vec{v}-\vec{v}_1\,'|^{\dagger}$$

The total change in the population of τ by collisions is then

$$\left(\frac{\partial F}{\partial t}\right)_C d^3r\,d^3v = -d^3r\,d^3v\int d^3v_1\,d\sigma\left[F(\vec{v})F(\vec{v}_1)-F(\vec{v})F(\vec{v}_1\,')\right]|\vec{v}-\vec{v}_1|$$

so that

$$\frac{dF}{dt}+\nabla\cdot(F\vec{v})+\nabla_{\vec{v}}\cdot(F\vec{a})=\int d^3v_1\,d\sigma\,|\vec{v}-\vec{v}_1|\left[F(\vec{v}\,')F(\vec{v}_1\,')-F(\vec{v})F(\vec{v}_1)\right]$$

---

† This follows from the fact that the Boltzmann Equation is invariant to space inversion and our assumption of a hard sphere model.

and when the interactions are weak so that $\vec{v}$ is not a function of $\vec{r}$, and $\vec{a}$ is either of the form

$$\vec{a} = \vec{a}(\vec{r}), \qquad \text{or} \qquad \vec{a} = \vec{v} \times \vec{B}(\vec{r})$$

then Boltzmann's Equation is of the form

$$\frac{dF}{dt} + \vec{v}\cdot\vec{\nabla}F + \vec{a}\,\vec{\nabla}_{\vec{v}}F = \int d\sigma\, d^3v_1 \left|\vec{v}-\vec{v}_1\right| \left[F'F_1' - FF_1\right].$$

I-5 Hydrodynamical Equations from Microscopic Point of View

A. Microscopic Transfer Equation

We shall derive the macroscopic hydrodynamical equations from microscopic point of view (i.e. by using the Boltzmann Equation), so that the classical hydrodynamical equations can be put on a more rigorous atomistic basis. We first define an average or expectation value of a function of velocity, space, and time, as

$$\langle \varphi(\vec{r}, \vec{v}, t) \rangle \equiv \frac{\int F \varphi\, d^3v}{\int F\, d^3v}.$$

We note that

$$n = \int F\, d^3v$$

so that

$$\langle \varphi(\vec{r}, \vec{v}, t) \rangle = \frac{1}{n} \int F \varphi\, d^3v$$

Now consider only those functions which depend on $\vec{v}$ alone. Then we have

$$\frac{\partial}{\partial t}\langle n\varphi\rangle = \frac{\partial}{\partial t}\int F\varphi(\vec{v})\,d^3v$$

$$= \int d^3v\,\varphi(\vec{v})\left[-\vec{v}\cdot\vec{\nabla}F - \vec{a}\cdot\vec{\nabla}_{\vec{v}}F - \left(\frac{\partial F}{\partial t}\right)_{Collision}\right]$$

$$= -\nabla\cdot\langle n\varphi(\vec{v})\vec{v}\rangle + \langle n\vec{a}\cdot\vec{\nabla}_{\vec{v}}\varphi(\vec{v})\rangle$$

$$+ \text{Collision term},$$

since

$$\int\varphi\vec{v}\cdot\vec{\nabla}F\,d^3v = \int d^3v\left[\nabla\cdot(F\varphi\vec{v}) - F\varphi\nabla\cdot\vec{v}\right] = \nabla\cdot\int F\varphi\vec{v}\,d^3v$$

$$\int\varphi\vec{a}\cdot\vec{\nabla}_{\vec{v}}F\,d^3v = \int d^3v\left[\nabla_{\vec{v}}\cdot(\varphi F\vec{a}) - F\vec{a}\cdot\frac{\partial\varphi}{\partial\vec{v}}\right] = \oint d\vec{S}_v\cdot(\varphi F\vec{a}) - \int F\vec{a}\cdot\frac{\partial\varphi}{\partial\vec{v}}\,d^3v = -\left\langle n\vec{a}\cdot\frac{\partial\phi}{\partial\vec{v}}\right\rangle.$$

It is shown in Reference (13) that the average value of the collision term can be put in the form

$$-\frac{1}{4}\int\left[\varphi+\varphi_1-\varphi'-\varphi_1'\right]\left[FF_1-F'F_1'\right]\left|\vec{v}-\vec{v}_1\right|\,d^3v\,d^3v_1\,d\sigma\,.$$

Thus

$$\frac{\partial}{\partial t}\langle n\varphi(\vec{v})\rangle + \nabla\cdot\langle n\varphi(\vec{v})\vec{v}\rangle - \langle n\vec{a}\cdot\vec{\nabla}_{\vec{v}}\varphi(\vec{v})\rangle$$

$$= \frac{1}{4}\int\left[\varphi+\varphi_1-\varphi'-\varphi_1'\right]\left[FF_1-F'F_1'\right]\left|\vec{v}-\vec{v}_1\right|\,d^3v\,d^3v_1\,d\sigma.$$

We note that

$$\varphi + \varphi_1 - \varphi' - \varphi_1'$$

vanishes if φ is m, $m\vec{v}$, or mv^2, since mass, momentum and energy are conserved in elastic collisions.

B. Macroscopic Variables

In order to describe the macroscopic variables in terms of the microscopic variables, we introduce

ρ = mass density $\equiv \langle n m \rangle$

$\vec{V}$ = fluid velocity $\equiv \dfrac{\langle n m \vec{v} \rangle}{\langle n m \rangle}$

$\vec{u}$ = thermal velocity $\equiv \vec{v} - \dfrac{\langle n m \vec{v} \rangle}{\langle n m \rangle}$

$$\langle \vec{u} \rangle = \langle \vec{v} - \vec{V} \rangle = \langle \vec{v} \rangle - \vec{V} = 0$$

$\overleftrightarrow{T}$ = stress tensor $\equiv \rho \langle u_i u_j \rangle$

$\vec{F}$ = external force density $\equiv \langle n m \vec{a} \rangle$

$n\epsilon$ = thermal energy density $\equiv n \langle \frac{1}{2} m u^2 \rangle$

Q_i = heat flow $\equiv \langle n \frac{1}{2} m u^2 u_i \rangle$

D_{ij} = deformation tensor $\equiv \frac{1}{2} \left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right]$

We shall deduce the hydrodynamical equations in terms of the above definitions.

1.) $\varphi = m \,, \quad \nabla_{\vec{v}} \varphi = 0$

$$\frac{\partial}{\partial t} \langle n \varphi \rangle = \frac{\partial \rho}{\partial t}, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

which is the equation of continuity.

2.) $$\varphi_i = m v_i \ , \quad \langle n\varphi_i \rangle = \langle n m v_i \rangle = \rho V_i$$

$$\vec{u} = \vec{v} - \vec{V} , \qquad \varphi_i = m v_i = m(u_i + V_i)$$

$$\nabla \cdot \langle n \varphi_i \vec{v} \rangle = \frac{\partial}{\partial x_j} \langle n m v_i v_j \rangle$$

$$= \frac{\partial}{\partial x_j} \langle nm\left[(u_i + V_i)(u_j + V_j) \right] \rangle$$

$$= \frac{\partial}{\partial x_j} \langle nm\left[u_i u_j + V_i V_j + u_i V_j + u_j V_i \right] \rangle$$

$$= \frac{\partial}{\partial x_j} \langle nm V_i V_j + nm u_i u_j \rangle$$

$$= \frac{\partial}{\partial x_j} \langle \rho V_i V_j \rangle + \frac{\partial}{\partial x_j} T_{ij} \ ,$$

since $\langle n m u_i V_j \rangle = 0$, $\langle n m u_j V_i \rangle = 0$, further

$$\vec{a} \cdot \vec{\nabla}_{\vec{v}} \varphi_i = a_j \frac{\partial \varphi_i}{\partial v_j} = a_j \frac{\partial}{\partial v_j} m v_i = m a_j \frac{\partial v_i}{\partial v_j} = m a_j \delta_{ij}$$

$$= m a_i = F_i / n$$

$$\frac{\partial}{\partial t} \rho V_i = -\frac{\partial}{\partial x_j} \left[T_{ij} + \rho V_i V_j \right] + F_i \ .$$

So using the equation of continuity, we have

$$\rho \dot{V}_i - V_j \frac{\partial}{\partial x_j} (\rho V_i) = -\frac{\partial T_{ij}}{\partial x_j} + F_i - \rho V_j \frac{\partial V_i}{\partial x_j} - V_i \frac{\partial}{\partial x_j} (\rho V_j)$$

or: $$\rho \left[\dot{V}_i + V_j \frac{\partial V_i}{\partial x_j} \right] = -\frac{\partial T_{ij}}{\partial x_j} + F_i$$

where $$\dot{V}_i \equiv \frac{\partial V_i}{\partial t}$$

or in vector notation, $\overleftrightarrow{T} = \hat{e}_i T_{ij} \hat{e}_j$, whence

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}\cdot\overleftrightarrow{T} + \vec{F}$$

This is the Stokes-Navier equation. When the stress tensor is diagonal,

$$T_{ij} = P\,\delta_{ij}$$

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \vec{F}.$$

3.) $\varphi = \frac{1}{2} m v^2, \quad \vec{v} = \vec{u} + \vec{V}, \quad \vec{V} = \frac{\langle nm\vec{v}\rangle}{\langle nm\rangle},$

$$\varphi = \frac{1}{2} m(\vec{u}+\vec{V})\cdot(\vec{u}+\vec{V}) = \frac{1}{2} m\left[u^2 + V^2 + 2\vec{u}\cdot\vec{V}\right]$$

$$= \frac{1}{2} m u^2 + \frac{1}{2} m V^2 + m\,\vec{u}\cdot\vec{V}$$

$$n\varphi v_i = \left(\frac{1}{2} nmu^2 + \frac{1}{2} nmV^2 + nm\,\vec{u}\cdot\vec{V}\right)(u_i + V_i)$$

$$= \frac{1}{2}\rho V^2 V_i + \frac{1}{2}\rho V^2 u_i + \frac{1}{2}\rho u^2 u_i + \frac{1}{2}\rho u^2 V_i$$

$$+ \rho u_j V_j u_i + \rho u_j V_j V_i$$

So that:

$$\langle n\varphi v_i\rangle = \frac{1}{2}\rho V^2 V_i + n\epsilon V_i + Q_i + T_{ij} V_j$$

and

$$\left\langle n\vec{a}\cdot\frac{\partial\varphi}{\partial\vec{v}}\right\rangle = \vec{F}\cdot\vec{V}$$

Whence:

$$\frac{\partial}{\partial t}\left[\frac{\rho V^2}{2} + n\epsilon\right] = -\frac{\partial}{\partial X_i}\left[\frac{1}{2}\rho V^2 V_i + T_{ij}V_j + Q_i + n\epsilon V_i\right] + \vec{F}\cdot\vec{V}$$

From the equation of motion and the equation of continuity we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2\right) = \dot{\rho}\frac{V^2}{2} + \rho\vec{V}\cdot\dot{\vec{V}}$$

$$= -\frac{V^2}{2}\left[\nabla\cdot(\rho\vec{V})\right] - \frac{\rho}{2}(\vec{V}\cdot\vec{\nabla})V^2 - \rho V\frac{\partial T_{ij}}{\partial X_j} + \vec{F}\cdot\vec{V}$$

and

$$\frac{\partial}{\partial X_i}\left(\frac{1}{2}\rho V^2 V_i\right) = \frac{1}{2}V^2\nabla\cdot(\rho\vec{V}) + \rho(\vec{V}\cdot\vec{\nabla})\frac{V^2}{2}$$

So that

$$\frac{\partial}{\partial t}(n\epsilon) = -T_{ij}\frac{\partial V_j}{\partial X_i} - \frac{\partial Q_i}{\partial X_i} - \frac{\partial}{\partial X_i}(n\epsilon V_i)$$

or

$$n\dot{\epsilon} + nV_i\frac{\partial \epsilon}{\partial X_i} = -T_{ij}D_{ij} - \frac{\partial Q_i}{\partial X_i}$$

i.e.

$$n\left[\frac{\partial \epsilon}{\partial t} + (\vec{V}\cdot\vec{\nabla})\epsilon\right] = -\overleftrightarrow{T}:\overleftrightarrow{D} - \vec{\nabla}\cdot\overleftrightarrow{Q}$$

This concludes our review of classical physics.

CHAPTER TWO

CLASSICAL STATISTICAL MECHANICS

II-1 Liouville's Theorem

We consider a dynamical system of f degrees of freedom, described by a conservative Hamiltonian

$$\mathcal{H} = \mathcal{H}(q_1, q_2, \cdots q_f ; p_1, p_2, \cdots p_f)$$

where $q_i ; p_i (i = 1,2,\ldots,f)$ are the generalized coordinates and momenta of the Hamiltonian. For instance, for a system containing n point molecules, $f = 3n$, and $q_i ; p_i$ are the ordinary position coordinates and linear momentum components. On the other hand, for a system of n vibrating rotating diatomic molecules, $f = (3+3)\, n = 6n$, owing to the two rotational degrees of freedom and one vibrational degree of freedom. Then the system is uniquely determined by Hamilton's equations of motion,

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad i = 1, 2, \cdots f. \qquad 1$$

Following Ehrenfest, the space of 2f dimensions D_{2f} defined by the $q_i ; p_i$ will be called the Γ-space or phase space. For example, there are 6n coordinates for the monoatomic gas, and 12n coordinates for the non-rigid diatomic gas. The instantaneous state of a system is represented by a point in this space, and the point describes a trajectory in the Γ-space corresponding to the instantaneous states of the dynamical system at each instant. That this trajectory does not intersect itself is evident from the fact that Hamilton's equation is of first order in time, so that there must be only single valued solutions and thus a unique trajectory through any point. The trajectories of any two points cannot intersect, because then they would be the same system at the point of intersection. Then each point has a velocity in this Γ-space, given by the 2f-dimensional vector:

$$\vec{v} = \vec{v}(\dot{q}_1, \dot{q}_2, \cdots \dot{q}_f ; \dot{p}_1, \dot{p}_2, \dot{p}_f)$$

which is defined at every point along its trajectory.

If we have a point in Γ-space, corresponding to a given system, then the neighborhood around this point includes points corresponding to systems which are very similar to the given system. For instance, in the case of the monoatomic gas they may be systems in which one molecule has been displaced by some distance δq or some momentum δp. Since these hypothetical displacements may be made as small as we wish, the number of systems within a given volume in the Γ-space may be made as large as desired, and consequently we may define a density of systems ρ which may be made arbitrarily large; but which is conserved, since the systems are not destroyed. Therefore, we have an equation of continuity in the Γ-space, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla_\Gamma \cdot (\rho \vec{v}) = 0 \qquad 2$$

where ρ = density of systems in Γ-space (i.e. ensemble density)

$$\vec{\nabla}_\Gamma = \vec{\nabla}_\Gamma\left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \cdots, \frac{\partial}{\partial q_f}; \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \cdots, \frac{\partial}{\partial p_f}\right)$$

$\vec{v}$ = Velocity of a point which determines the instantaneous state of a system in Γ-space

$$= \vec{v}(\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_f; \dot{p}_1, \dot{p}_2, \cdots, \dot{p}_f)$$

$$\vec{\nabla}_\Gamma \cdot \vec{v} = \sum_{i=1}^{f}\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) = \sum_{i=1}^{f}\left(\frac{\partial}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right)$$

$$= \sum_{i=1}^{f}\left(\frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_i} - \frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_i}\right) = 0 \qquad \therefore \vec{\nabla}_\Gamma \cdot (\rho \vec{v}) = \vec{v} \cdot \vec{\nabla}_\Gamma \rho$$

whence

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_\Gamma\right)\rho = \frac{d\rho}{dt} = 0 \qquad 3$$

But $\frac{d\rho}{dt}$ is just the rate of change of density in a fluid that moves at a velocity $\vec{v}$. Thus the motion of the phase points in the Γ-space is like that of an incompressible fluid, since

$$\int_\Gamma \vec{\nabla}_\Gamma \cdot \vec{v}\, d\tau = \oint_\Gamma \vec{v} \cdot d\vec{S}$$

and

$$\vec{v} \cdot d\vec{S}_\Gamma = \frac{d\tau}{dt}, \qquad \therefore \frac{d\tau}{dt} = 0$$

where

$$d\tau \equiv dq_1\, dq_2 \cdots dq_f\, dp_1\, dp_2 \cdots dp_f$$
$$= \prod_{i=1}^{f} dq_i\, dp_i = \prod_{i=1}^{f} d^6 \zeta_i$$

Thus the volume τ occupied by the phase points in Γ-space does not change. The analogy with an incompressible fluid can be carried further by observing that the shape of the volume may change. That this is obvious can be seen from the example of two systems, the molecules of one all moving in a parallel direction at a uniform velocity, and the other having one molecular velocity displaced through a small angle. If there are perpendicular reflecting walls, the first system will continue in the same pattern after reflected from the wall, while the second system will have intermolecular collisions and finally reach some equilibrium configuration. But no matter how the shape may change the volume encompassed by a group of phase points is constant.

In terms of the Poisson bracket defined in Section I-B, Liouville's theorem can be put in a more convenient form:

$$\frac{\partial \rho}{\partial t} + [\rho, \mathcal{H}] = \frac{d\rho}{dt} = 0 \qquad 4$$

since

$$\begin{aligned}\vec{v} \cdot \vec{\nabla}_\Gamma \, \rho &= \sum_{i=1}^{f} \dot{q}_i \frac{\partial \rho}{\partial q_i} + \sum_{i=1}^{f} \dot{p}_i \frac{\partial \rho}{\partial p_i} \\ &= \sum_{i=1}^{f} \left[\frac{\partial \rho}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right] \\ &= [\rho, \mathcal{H}]\end{aligned}$$

$$\rho = \rho(q_1, q_2, \cdots, q_f ; p_1, p_2, \cdots p_f)$$

II-2 The H-Theorem and Canonical Ensemble

A fundamental result of kinetic theory and statistical mechanics is the Boltzmann H-theorem. For a classical system the H-theorem† can be put in the form

$$\frac{dH}{dt} \leqslant 0$$

where

$$H = \int \prod_{i=1}^{f} d^6\zeta_i \, \rho \ln \rho \ ,$$

$$\rho = \rho(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_f ; \vec{p}_1, \vec{p}_2, \cdots \vec{p}_f ; t)$$

is the ensemble density function in Γ-space, and

$$\prod_{i=1}^{f} d^6\zeta_i = \prod_{i=1}^{f} d^3 r_i \, d^3 p_i$$

is the volume element in Γ-space.

The H-theorem can be stated as follows:

Subject to whatever constraints which may be imposed on the system, the quantity H approaches a minimum value in the passage of time and remains at that value for an undisturbed system. The system is then in statistical, or thermodynamic, equilibrium.

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† For a rigorous treatment for the H-theorem see, for example, Reference 6.

We shall apply the H-theorem to find the equilibrium of the Gibbs canonical ensemble.

The canonical ensemble is defined by

$$\int \mathcal{H} \rho \prod_{i=1}^{f} d^6 \zeta_i = E = Constant$$

(where $\mathcal{H}$ is the typical system Hamiltonian), i.e., the average energy is constant.

The systems of the ensemble may be thought of as interacting very slightly with each other so as to be able to exchange energy in such a way that the energy of the entire ensemble remains constant.

We now minimize H subject to

$$\delta E = \int \delta\rho \, \mathcal{H} \prod_{i=1}^{f} d^6 \zeta_i = 0 \qquad 3a$$

$$\delta \int \rho \prod_{i=1}^{f} d^6 \zeta_i = \delta(1) = 0 \qquad 3b$$

If Eq. 3a is multiplied by the Lagrange multiplier $-\left(1+\frac{\psi}{\Theta}\right)$, and Eq. 3b by $\frac{1}{\Theta}$, and these are added to the variation of H, i.e.,

$$\delta H = \int \delta\rho \left[\ln\rho + 1 \right] \prod_{i=1}^{f} d^6 \zeta_i = 0 ,$$

the result is

$$\int \delta\rho \left[\ln\rho - \frac{\psi - \mathcal{H}}{\Theta} \right] \prod_{i=1}^{f} d^6 \zeta_i = 0$$

or

$$\rho = e^{\frac{\Psi - \mathcal{H}}{\Theta}}$$

and so,

$$\mathrm{H} = \int \rho \ln \rho \prod_{i=1}^{f} d^6\zeta_i = \int exp\left(\frac{\Psi-\mathcal{H}}{\Theta}\right) \ln exp\left(\frac{\Psi-\mathcal{H}}{\Theta}\right) \prod_{i=1}^{f} d^6\zeta_i$$

$$= \frac{\Psi}{\Theta}\int exp\left(\frac{\Psi-\mathcal{H}}{\Theta}\right)\prod_{i=1}^{f} d^6\zeta_i - \frac{1}{\Theta}\int \mathcal{H}\, exp\left(\frac{\Psi-\mathcal{H}}{\Theta}\right)\prod_{i=1}^{f} d^6\zeta_i$$

$$= \frac{\Psi}{\Theta}\int \rho \prod_{i=1}^{f} d^6\zeta_i - \frac{1}{\Theta}\int \mathcal{H}\,\rho \prod_{i=1}^{f} d^6\zeta_i$$

$$= \frac{\Psi - E}{\Theta} \qquad 4$$

Consequently

$$\delta\mathrm{H} = \delta\left(\frac{\Psi - E}{\Theta}\right) = \frac{\delta\Psi - \delta E}{\Theta} - \frac{\Psi - E}{\Theta^2}\delta\Theta \qquad 5$$

and since

$$\delta\int \prod_{i=1}^{f} d^6\zeta_i\, \rho = \delta(1) = 0 = \int \prod_{i=1}^{f} d^6\zeta_i\, \delta\left[exp\left(\frac{\Psi-\mathcal{H}}{\Theta}\right)\right]$$

or

$$\int e^{\frac{\Psi-\mathcal{H}}{\Theta}}\left[\frac{\delta\Psi - \delta\mathcal{H}}{\Theta} - \frac{\Psi-\mathcal{H}}{\Theta^2}\delta\Theta\right]\prod_{i=1}^{f} d^6\zeta_i = 0,$$

we have

$$\frac{\delta\Psi}{\Theta} - \frac{\Psi}{\Theta^2}\delta\Theta + \frac{\delta\Theta}{\Theta^2}\int \rho \mathscr{H} \prod_{i=1}^{f} d^6\zeta_i - \frac{1}{\Theta}\int \rho\,\delta\mathscr{H} \prod_{i=1}^{f} d^6\zeta_i = 0,$$

or

$$\frac{\delta\Psi}{\Theta} - \frac{\Psi - E}{\Theta^2}\delta\Theta + \frac{\delta W}{\Theta} = 0 \qquad 6$$

where $-\delta W = \int \rho\,\delta\mathscr{H} \prod_{i=1}^{f} d^6\zeta_i$ is the work done on the system. Combining Eqs. (5) and (6),

$$-\delta H = \frac{\delta E + \delta W}{\Theta} \qquad 7$$

We identify $-kH$ with the entropy of the system, Θ/k with the temperature T, and Ψ with the free energy. Then Eq. (7) becomes

$$-k\,\delta H = \frac{\delta E + \delta W}{T}$$

i.e.

$$\delta S = \frac{\delta E + \delta W}{T}, \qquad S = -kH$$

or

$$\delta Q = T\delta S = \delta E + \delta W.$$

Also, $$H = \frac{\Psi - E}{\Theta} \qquad \text{(from eq. 4)}$$

$$-kH = k\frac{E - \Psi}{kT} = \frac{E - \Psi}{T}$$

$$S = -kH = \frac{E - \Psi}{T}$$

and

$$\psi = E - T S, \quad d\psi = (T dS - \delta W) - T dS - S dT$$

$$d\psi = -\delta W - S dT = -P dV - S dT \quad (\delta Q = T dS,\ \delta W = P dV)$$

$$\therefore \quad P = -\left(\frac{\partial \psi}{\partial V}\right)_T, \quad S = -\left(\frac{\partial \psi}{\partial T}\right)_V .$$

From the H-theorem, we thus have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \leqslant 0,$$

and since

$$S = -kH$$

$$\therefore \quad \frac{dS}{dt} = \frac{\partial S}{\partial t} \geqslant 0$$

The H-theorem is then a statement of the thermodynamical law that for an isolated system the entropy tends to a maximum value at equilibrium, and that the entropy of the universe always increases for irreversible processes.

CHAPTER THREE

THE DYNAMICS OF MANY-PARTICLE SYSTEMS

III-1 The Method of Self-Consistent Fields

It is well known in atomic physics that an excellent approximation for solving the many electron atoms is to describe the motion of a given electron in the atom by assuming that the forces experienced by this electron are due to the averaging Coulomb fields of the nucleus and all the other electrons. It is physically reasonable to expect that a similar method can be applied to a plasma. Thus, the motion of a given particle in the plasma may be obtained by considering the forces experienced by this particle due to the externally applied electromagnetic fields plus the microscopically smoothed fields due to the motions of all of the other nearby particles in the plasma. This means that instead of having to solve for the precise orbits of all of the particles in each other's fluctuating force fields, we need to solve only for the motion of a typical particle in the microscopically smoothed electromagnetic fields. The central problem of this approach is to find the partial differential equations (the Boltzmann-Vlasov equations) that describe the evolution of the distribution function of a typical particle in time and that directly involves the smoothed fields. This distribution function is then formally used to compute the microscopic charge and current densities present in the plasma. These charge and current densities (which are already functionals of the electromagnetic fields) are then inserted into Maxwell's equations to yield a "self-consistent" solution. The resulting mathematical problem is clearly non-linear and very approximate methods of solving even this self-consistent approximation must be used.

This single particle distribution function contains all the possible information about the dynamical behavior of the system since differential equations that involve only physically interesting microscopic observables may be obtained from the Boltzmann-Vlasov equations in a way similar to that used for deriving the classical hydrodynamical equations from the Boltzmann transport equation. However, a closed set of equations cannot be obtained in this way. For if we take a certain moment of the Boltzmann-Vlasov equations in order to find out how a given quantity develops in time, we find invariably that this evolution in time depends on another higher moment. The customary practice has been to close this sequence of equations somewhat arbitrarily. The introduction of a set of transport coefficients and certain phenomenological relations (such as Ohm's Law) combined with the moment equations will lead to a set of equations known as the magnetohydrodynamical equations which are still not too easy to solve in most cases of interest.

III-2 Hamilton's Equation for a Charged Particle in External Fields

The Hamiltonian for a relativistic charged particle in an external field derivable from a scalar and vector potential can be written as

$$\mathcal{H} = c\sqrt{m_0^2c^2 + (\vec{p} - \frac{e}{c}\vec{A})^2} + e\varphi \qquad 1$$

where $\vec{p}$ is the (generalized) momentum, m_0 is the rest mass of the particle, and

e = charge carried by the particle

c = speed of light

φ = the scalar potential of the external field

$\vec{A}$ = the vector potential of the external field

The Hamilton's equations of motion are

$$\dot{\vec{r}} = \frac{\partial \mathcal{H}}{\partial \vec{p}}, \quad \dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{r}} \qquad 2$$

On making use of Eq. 1, the first of equations 2 becomes

$$\frac{d\vec{r}}{dt} = \frac{\partial}{\partial \vec{p}}\left\{c\left[m_0^2c^2 + (\vec{p} - \frac{e}{c}\vec{A})^2\right]^{1/2} + e\varphi\right\}$$

$$= \frac{c(\vec{p} - \frac{e}{c}\vec{A})}{\sqrt{m_0^2c^2 + (\vec{p} - \frac{e}{c}\vec{A})^2}},$$

since

$$\frac{\partial}{\partial p_j}(p_i - \frac{e}{c}A_i)^2 = 2(p_i - \frac{e}{c}A_i)\delta_{ij}.$$

It follows that

$$\vec{p} = \gamma m_0 \vec{v} + \frac{e}{c}\vec{A} \qquad 3$$

where

$$\vec{v} \equiv \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \vec{\beta} = \frac{\vec{v}}{c}, \quad \beta = \frac{v}{c}.$$

Now

$$\vec{\nabla}\mathcal{H} = \frac{\partial\mathcal{H}}{\partial\vec{r}} = c\vec{\nabla}\varphi - \frac{c\vec{\nabla}(\vec{p}-\frac{e}{c}\vec{A})^2}{2\sqrt{m_o^2c^2+(\vec{p}-\frac{e}{c}\vec{A})^2}},$$

and

$$\vec{\nabla}(\vec{p}-\frac{e}{c}\vec{A})^2 = 2\left\{(\vec{p}-\frac{e}{c}\vec{A})\times\vec{\nabla}\times(\vec{p}-\frac{e}{c}\vec{A}) + \left[(\vec{p}-\frac{e}{c}\vec{A})\cdot\vec{\nabla}\right](\vec{p}-\frac{e}{c}\vec{A})\right\},$$

using the vector identity

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{B}\times\vec{\nabla}\times\vec{A} + \vec{A}\times\vec{\nabla}\times\vec{B}.$$

We therefore have

$$\vec{\nabla}(\vec{p}-\frac{e}{c}\vec{A})^2 = 2\gamma m_o\vec{v}\times\left[\vec{\nabla}\times(\vec{p}-\frac{e}{c}\vec{A})\right] + 2\gamma m_o\vec{v}\cdot\vec{\nabla}(\vec{p}-\frac{e}{c}\vec{A})$$

$$= 2\gamma m_o\vec{v}\times\left[-\frac{e}{c}\vec{\nabla}\times\vec{A}\right] + 2\gamma m_o\vec{v}\cdot\vec{\nabla}(-\frac{e}{c}\vec{A})$$

$$= -\frac{2\gamma m_o e}{c}\left[\vec{v}\times\vec{B} + (\vec{v}\cdot\vec{\nabla})\vec{A}\right]$$

because

$$\vec{B} = \vec{\nabla}\times\vec{A}, \quad \vec{\nabla}\times\vec{p} = 0, \quad (\vec{v}\cdot\vec{\nabla})\vec{p} = 0.$$

Therefore the second of equations 2 gives

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\frac{\partial\vec{A}}{\partial t} + \frac{c}{2\gamma m_o c}2\frac{\gamma m_o e}{c}\left[\vec{v}\times\vec{B} + (\vec{v}\cdot\vec{\nabla})\vec{A}\right]$$

$$= e\vec{E} + \frac{e}{c}\frac{\partial\vec{A}}{\partial t} + \frac{e}{c}\vec{v}\times\vec{B} + \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A}$$

$$= e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} + \frac{e}{c}\frac{d\vec{A}}{dt} \qquad 4$$

since

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} .$$

From equation 3 we have

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\frac{d\vec{A}}{dt} + \gamma m_0 \frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt} \qquad 5$$

Combine equations 4 and 5:

$$e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} + \frac{e}{c}\frac{d\vec{A}}{dt} = \frac{e}{c}\frac{d\vec{A}}{dt} + \gamma m_0\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt} ,$$

or

$$e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} = \gamma m_0\frac{d\vec{v}}{dt} + m_0\vec{v}\frac{d\gamma}{dt} \qquad 5'$$

Take the dot product of 5' by $\vec{v}$, and we get

$$e\vec{v}\cdot\vec{E} + 0 = \gamma m_0\vec{v}\cdot\frac{d\vec{v}}{dt} + m_0\vec{v}\,\vec{v}\frac{d\gamma}{dt}$$

$$= \frac{1}{2}m_0\gamma\frac{dv^2}{dt} + m_0 v^2\frac{d\gamma}{dt} .$$

Now

$$\frac{d\gamma}{dt} = -\frac{1}{2}(1-\beta^2)^{-3/2}(-2\beta\dot{\beta})$$

$$= \gamma^3\beta\dot{\beta} = \gamma^3\frac{\vec{v}\cdot\dot{\vec{v}}}{c^2}$$

and

$$
\begin{aligned}
e\vec{v}\cdot\vec{E} &= \tfrac{1}{2}\gamma m_o \frac{dv^2}{dt} + m_o v^2 \frac{\gamma^3}{c^2}\dot{\vec{v}}\cdot\vec{v} \\
&= (1+\gamma^2\beta^2)\,\tfrac{1}{2} m_o \gamma \frac{dv^2}{dt} \\
&= \tfrac{1}{2} m_o \gamma^3 \frac{dv^2}{dt}{}^{\dagger} = m_o \gamma^3 \vec{v}\cdot\dot{\vec{v}},
\end{aligned}
$$

and so

$$\frac{e}{c^2}\vec{v}\cdot\vec{E} = m_o\gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}.$$

Thus Eq. 5' becomes

$$
\begin{aligned}
e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} &= \gamma m_o \frac{d\vec{v}}{dt} + m_o \vec{v}\gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}}, \\
e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} - \frac{e}{c^2}\vec{v}(\vec{v}\cdot\vec{E}) &= \gamma m_o \frac{d\vec{v}}{dt}, \\
\frac{m_o}{\sqrt{1-\beta^2}}\frac{d\vec{v}}{dt} &= e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} - \frac{e}{c^2}(\vec{v}\cdot\vec{E})\vec{v}
\end{aligned}
$$

and so

$$m_o\frac{d\vec{v}}{dt} = \sqrt{1-\frac{v^2}{c^2}}\left[e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} - \frac{e}{c^2}(\vec{v}\cdot\vec{E})\vec{v}\right] \qquad 6$$

which is the Lorentz-Newton equation of motion in relativistic form. In the non-relativistic limit, i.e., neglecting β^2 and higher order

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† Since $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$, $1+\gamma^2\beta^2 = \gamma^2$.

terms, Eq. 6 becomes the familiar Lorentz force equation:

$$m_0 \frac{d\vec{v}}{dt} = e\left[\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right] \qquad 7$$

III-3 Liouville's Equation for a Relativistic Charged Particle in an External Field

The Liouville equation is

$$\frac{\partial f}{\partial t} + [\,f, \mathcal{H}\,] = 0, \quad f = f(\vec{r}, \vec{p}, t) \qquad 1$$

where f is the distribution function or density function, i.e. the function ρ defined before.†

For a single particle the Poisson bracket becomes

$$[\,f, \mathcal{H}\,] = \sum_{i=1}^{3}\left\{\frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right\} = \frac{\partial f}{\partial \vec{r}}\cdot\frac{\partial \mathcal{H}}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}}\cdot\frac{\partial \mathcal{H}}{\partial \vec{r}} \qquad 2$$

Now

$$\frac{\partial \mathcal{H}}{\partial \vec{p}} = \vec{\nabla}_{\vec{p}}\,\mathcal{H} = \frac{c(\vec{p} - \frac{e}{c}\vec{A})}{\sqrt{m_0^2 c^2 + (\vec{p} - \frac{e}{c}\vec{A})^2}} = \frac{c(\vec{p}\ \ \frac{e}{c}\vec{A})}{\gamma m_0 c} = \frac{\vec{P}}{\gamma m_0} = \vec{v} \qquad 3$$

where

$$\vec{P} = \vec{p} - \frac{e}{c}\vec{A} = \gamma\, m_0 \vec{v}\ ^{\S} \qquad 4$$

($\vec{P}$ = mechanical momentum of the particle)

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† ρ is the many particle distribution, but f is a single particle distribution function.

§ $\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}}$, $\mathcal{L} = -m_0 c^2 \sqrt{1 - \frac{\dot{r}^2}{c^2}} + \frac{e}{c}\vec{A}\cdot\dot{\vec{r}} - e\varphi$

$\vec{P} = \gamma m_0 \vec{v}$ is the mechanical momentum in the absence of fields.

$$-\frac{\partial \mathcal{H}}{\partial \vec{r}} = e\vec{E} + \frac{e}{c}\frac{\partial \vec{A}}{\partial t} + \frac{e}{c}\vec{v}\times\vec{B} + \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A},$$

$$[f,\mathcal{H}] = \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \left[e\vec{E} + \frac{e}{c}\frac{d\vec{A}}{dt} + \frac{e}{c}\vec{v}\times\vec{B}\right]\cdot\frac{\partial f}{\partial \vec{p}} \qquad 5$$

Since

$$\dot{\vec{r}} = \vec{v} = \frac{\partial \mathcal{H}}{\partial \vec{p}},$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{r}}$$

where

$$f = f(\vec{r}, \vec{p}, t)$$

We now transform the independent variables according to

$$\vec{r},\ \vec{p},\ t \longrightarrow \vec{r},\ \vec{P},\ t$$

where

$$\vec{P} = \vec{p} - \frac{e}{c}\vec{A}(\vec{r},t) \quad \text{as before.}$$

Then

$$\frac{\partial}{\partial \vec{p}} \longrightarrow \frac{\partial}{\partial \vec{P}}, \quad \frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} - \frac{e}{c}\frac{\partial \vec{A}}{\partial t}\frac{\partial}{\partial \vec{P}}$$

$$\vec{v}\frac{\partial}{\partial \vec{r}} \longrightarrow \vec{v}\frac{\partial}{\partial \vec{r}} - \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A}\cdot\frac{\partial}{\partial \vec{P}} \qquad 6$$

Carrying out the indicated change of variables, we have

$$\frac{\partial f}{\partial t} - \frac{e}{c}\frac{\partial \vec{A}}{\partial t}\cdot\frac{\partial f}{\partial \vec{P}} + \vec{v}\cdot\vec{\nabla}f - \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A}\,\frac{\partial f}{\partial \vec{P}}$$

$$+\left[e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}\right]\cdot\frac{\partial f}{\partial \vec{P}} + \frac{e}{c}\frac{d\vec{A}}{dt}\cdot\frac{\partial f}{\partial \vec{P}} = 0,$$

$$\frac{\partial f}{\partial t} + \frac{1}{\gamma m_0}\vec{P}\cdot\vec{\nabla}f + \left[e\vec{E} + \frac{1}{\gamma m_0}\vec{P}\times\vec{B}\right]\cdot\frac{\partial f}{\partial \vec{P}}$$

$$+\left[\frac{e}{c}\left(\frac{d\vec{A}}{dt} - \frac{\partial \vec{A}}{\partial t} - \vec{v}\cdot\vec{\nabla}\vec{A}\right)\right]\cdot\frac{\partial f}{\partial \vec{P}} = 0,$$

i.e.

$$\frac{\partial f}{\partial t}+\frac{1}{\gamma m_0}\vec{P}\cdot\vec{\nabla}f+e\left[\vec{E}+\frac{\vec{P}\times\vec{B}}{\sqrt{P^2+m_0^2c^2}}\right]\cdot\frac{\partial f}{\partial\vec{P}}=0, \qquad 7$$

or

$$\frac{\partial f}{\partial t}+\frac{c\vec{P}}{\sqrt{P^2+m_0^2c^2}}\cdot\frac{\partial f}{\partial\vec{r}}+e\left[\vec{E}+\frac{\vec{P}\times\vec{B}}{\sqrt{P^2+m_0^2c^2}}\right]\cdot\frac{\partial f}{\partial\vec{P}}=0$$

$$\because \quad \frac{d\vec{A}}{dt}=\frac{\partial\vec{A}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{A}$$

$$\sqrt{P^2+m_0^2c^2}=\gamma m_0 c\,, \quad \gamma=\frac{1}{\sqrt{1-\beta^2}}$$

Eq. 7 is the desired Liouville equation in relativistic form. This equation may be interpreted as the Boltzmann-Vlasov† equation provided we interpret the single particle distribution function in the self-consistent field approximation, and the fields $\vec{E}$ and $\vec{B}$ represent the sums of the external fields and the solution of Maxwell's equations with charge and current densities expressed in terms of the distribution function $f(\vec{r},\vec{P},t)$ by the formulas

$$\begin{aligned}\rho_e &= Ne\int d^3P\, f(\vec{r},\vec{P},t)\\ \vec{J} &= Ne\int d^3P\,\frac{1}{\sqrt{P^2+m_0^2c^2}}\vec{P}\, f(\vec{r},\vec{P},t)\end{aligned} \qquad 8$$

We note that it is equation 7 with the interpretation of $\vec{B}$ and $\vec{E}$ implied by equation 8 that will be called the relativistic form of the B-V equation.

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† A rigorous demonstration of this assertion will be given in Chapter IV-2.

We shall return to the detailed proof in Chapter IV for the equivalence of the B-V equation and the Liouville equation for a single particle in the self-consistent treatment.

We now transform Eq. 7 from $\vec{r}, \vec{P}, t$ to $\vec{r}\ \vec{v}\ t$. The transformation is clearly given by

$$\frac{\partial}{\partial \vec{r}} \longrightarrow \frac{\partial}{\partial \vec{r}}, \quad \frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \vec{p}} \longrightarrow \frac{\partial \vec{v}}{\partial \vec{p}} \frac{\partial}{\partial \vec{v}}$$

$$\frac{\partial v_j}{\partial P_i} = \frac{\partial}{\partial P_i}\left[\frac{c P_j}{\sqrt{m_0^2 c^2 + P^2}}\right] = \frac{c}{\sqrt{m_0^2 c^2 + p^2}}\, \delta_{ij} + c P_j \frac{\partial}{\partial P_i}(m_0^2 c^2 + p^2)^{-1/2}$$

$$= \frac{c}{\sqrt{m_0^2 c^2 + p^2}}\, \delta_{ij} - \frac{1}{2} c P_j [m_0^2 c^2 + p^2]^{-3/2} \frac{\partial P^2}{\partial P_i}$$

$$= \frac{c}{\sqrt{m_0^2 c^2 + p^2}}\, \delta_{ij} - \frac{1}{2} c P_j \frac{2 P_i \delta_{ij}}{(\sqrt{m_0^2 c^2 + p^2})^3}$$

$$= \frac{c}{\sqrt{m_0^2 c^2 + p^2}}\, \delta_{ij} - \frac{c P_i P_j}{(\sqrt{m_0^2 c^2 + p^2})^3}\, \delta_{ij}$$

$$= \frac{c}{\sqrt{m_0^2 c^2 + p^2}}\, \delta_{ij}\left[1 - \frac{P_i P_j}{m_0^2 c^2 + p^2}\right]$$

$$= \frac{c}{\gamma m_0 c}\, \delta_{ij}\left[1 - \frac{P_i P_j}{\gamma^2 m_0^2 c^2}\right]$$

$$= \frac{1}{\gamma m_0}\, \delta_{ij}\left[1 - \frac{(\gamma m_0 v_i)(\gamma m_0 v_j)}{(\gamma m_0 c)^2}\right]$$

$$= \frac{1}{\gamma m_0}\, \delta_{ij}\left[1 - \frac{v_i v_j}{c^2}\right]$$

$$= \frac{1}{m_0}\, \delta_{ij} \sqrt{1 - \frac{v^2}{c^2}}\left[1 - \frac{v_i v_j}{c^2}\right]$$

$$\frac{\partial v_j}{\partial P_j} = \frac{1}{m_0} \sqrt{1 - \frac{v^2}{c^2}}\left[1 - \frac{v_j v_j}{c^2}\right]$$

$$\frac{\partial}{\partial \vec{p}} \longrightarrow \frac{\partial \vec{v}}{\partial \vec{p}} \frac{\partial}{\partial \vec{v}} , \quad \frac{\partial}{\partial p_i} \rightarrow \frac{\partial v_j}{\partial p_i} \frac{\partial}{\partial v_j}$$

$$\frac{\partial}{\partial p_i} \longrightarrow \frac{1}{m_0}\sqrt{1-\beta^2}\ \delta_{ij}\left[1 - \frac{v_i v_j}{c^2}\right] \frac{\partial}{\partial v_j} = \frac{1}{\gamma m_0}\left[1-\frac{v_i}{c^2}v_i\right]\frac{\partial}{\partial v_i}$$

$$\frac{\partial}{\partial p_j} \longrightarrow \frac{1}{m_0}\sqrt{1-\beta^2}\left[1-\frac{v_j v_j}{c^2}\right]\frac{\partial}{\partial v_j} \qquad 9$$

$$\therefore \quad \frac{\partial}{\partial \vec{r}} \longrightarrow \frac{\partial}{\partial \vec{r}}, \quad \frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t},$$

$$\frac{\partial}{\partial \vec{P}} \longrightarrow \frac{1}{\gamma m_0}\left[1-\frac{\vec{v}\vec{v}}{c^2}\right]\cdot\frac{\partial}{\partial \vec{v}}, \quad \vec{P} = \gamma m_0 \vec{v} = \vec{p} - \frac{e}{c}\vec{A}$$

and so

$$\frac{\partial f}{\partial t} + \frac{c}{\gamma m_0 c}(\gamma m_0 \vec{v})\cdot\frac{\partial f}{\partial \vec{r}} + e\left[\vec{E} + \frac{1}{\gamma m_0 c}\gamma m_0 \vec{v}\times\vec{B}\right]\cdot\frac{1}{\gamma m_0}\left[\frac{\partial f}{\partial \vec{v}} - \frac{\vec{v}}{c^2}\left(\vec{v}\cdot\frac{\partial f}{\partial \vec{v}}\right)\right]$$
$$= 0$$

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + e\left[\frac{1}{\gamma m_0}\vec{E}\cdot\frac{\partial f}{\partial \vec{v}} - \frac{1}{\gamma m_0}\frac{1}{c^2}(\vec{E}\cdot\vec{v})\left(\vec{v}\cdot\frac{\partial f}{\partial \vec{v}}\right) + \frac{1}{\gamma m_0 c}\vec{v}\times\vec{B}\cdot\frac{\partial f}{\partial \vec{v}}\right.$$
$$= 0$$

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{e}{m_0}\sqrt{1-\frac{v^2}{c^2}}\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B} - \frac{\vec{v}\cdot\vec{E}}{c^2}\vec{v}\right]\cdot\vec{\nabla}_{\vec{v}} f = 0$$

$$10$$

In the non-relativistic limit, we get

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{e}{m_0}\left[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right]\cdot\vec{\nabla}_{\vec{v}} f = 0 \ .$$

III-4 Hamilton's Equations for a Many-Particle System

The Hamiltonian for a many-particle system subjected to external fields represented by $\vec{A}(\vec{r},t)$ and $\varphi(\vec{r},t)$ is given to order $\frac{v^2}{c^2}$ by the Darwin Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \frac{1}{2m_i}\left[\vec{p}_i - \frac{e_i}{c}\vec{A}(\vec{r}_i,t)\right]^2 + \sum_{i=1}^{N} e_i\varphi(\vec{r}_i,t)$$
$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}{}' \frac{e_i e_j}{|\vec{r}_i-\vec{r}_j|} - \frac{1}{2c^2}\sum_{i=1}^{N}\sum_{j=1}^{N}{}' \frac{e_i e_j}{2m_i m_j}\left[\frac{\vec{p}_i\cdot\vec{p}_j}{|\vec{r}_i-\vec{r}_j|} + \frac{\vec{p}_i\cdot\vec{r}_{ij}\,\vec{p}_j\cdot\vec{r}_{ij}}{|\vec{r}_i-\vec{r}_j|^3}\right]$$
$$-\sum_{i=1}^{N} \frac{p_i^4}{8m_i^3c^2}$$

where $\vec{p}_i$, $\vec{r}_i$ are the momenta and coordinates of the ith particle and $\vec{r}_{ij}=\vec{r}_i-\vec{r}_j$. The meanings of the first three terms are obvious. The fourth term represents the magnetic interactions between the particles. We note that this term, which is of order $\frac{v^2}{c^2}$ compared to the other terms, cannot be omitted because the effects produced by this term may actually be quite appreciable if the particles behave in a collective way so that, for instance, large induced magnetic fields are produced. In this case the expansion parameter is not actually the quantity $\frac{v^2}{c^2}$, but rather this quantity multiplied by the number of particles. Therefore, for many systems of interest the magnetic interaction term may play a much more important role than the Coulomb interaction term. This would happen, for example, if the particles are distributed with uniform density so that on the average the Coulomb repulsions cancel each other out.

Hamilton's equations for this many-particle system are

$$\frac{d\vec{r}_i}{dt} = \frac{\partial \mathcal{H}}{\partial \vec{p}_i}, \quad \frac{d\vec{p}_i}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{r}_i}, \quad i = 1, 2, \cdots, N$$

We shall rewrite the Hamiltonian as follows:†

$$\mathcal{H} = K + V$$

where

$$K = \frac{1}{2m}\sum_i \left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right)^2 + \sum_i e\varphi_i$$

and

$$V = \frac{1}{2}\sum_i \sum_j{}' \frac{e^2}{r_{ij}} - \frac{1}{2}\frac{e^2}{2m^2c^2}\sum_i \sum_j \left\{ \frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}} + \frac{\vec{p}_i \cdot \vec{r}_{ij}\, \vec{p}_j \cdot \vec{r}_{ij}}{r_{ij}^3} \right\}$$

It is clear that K is the interaction energy of the individual particles with the external fields, and V is the interaction energy between the particles, i.e., Coulomb interaction and magnetic interaction.

To proceed further, we prove the following lemmas:

Lemma 1°

$$\frac{\partial V_{ij}}{\partial \vec{p}_i} = 2\left[\frac{\vec{p}_j}{|\vec{r}_i - \vec{r}_j|} - \frac{1}{2}\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i}|\vec{r}_i - \vec{r}_j| \right) \right]$$

where

$$V_{ij} = \frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}} + \frac{\vec{p}_i \cdot \vec{r}_{ij}\, \vec{p}_j \cdot \vec{r}_{ij}}{r_{ij}^3}, \quad r_{ij} = |\vec{r}_i - \vec{r}_j|$$

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† For simplicity the last terms in the Darwin Hamiltonian, which is purely relativistic, has been neglected and we have set $m_i = m$, $e_i = e$.

Proof:

$$\frac{\partial}{\partial \vec{r}_i}\left[\vec{p}_j\cdot\frac{\partial r_{ij}}{\partial \vec{r}_i}\right]=\vec{\nabla}_i\left[\vec{p}_j\cdot\left(\frac{\vec{r}_{ij}}{r_{ij}}\right)\right]=\vec{\nabla}_i\left[(\vec{p}_j\cdot\vec{r}_{ij})\left(\frac{1}{r_{ij}}\right)\right]$$

$$=\frac{1}{r_{ij}}\vec{\nabla}_i(\vec{p}_j\cdot\vec{r}_{ij})+(\vec{p}_j\cdot\vec{r}_{ij})\vec{\nabla}_i\frac{1}{r_{ij}}=\frac{\vec{p}_j}{r_{ij}}-\frac{\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^3}\vec{r}_{ij}$$

$$\frac{\partial V_{ij}}{\partial \vec{p}_i}=\vec{\nabla}_{p_i}\left[\frac{\vec{p}_i\cdot\vec{p}_j}{r_{ij}}+\frac{\vec{p}_i\cdot\vec{r}_{ij}\,\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^3}\right]=\frac{\vec{p}_j}{r_{ij}}+\frac{\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^{3}}\vec{r}_{ij}$$

$$=\frac{\vec{p}_j}{r_{ij}}+\left[\frac{\vec{p}_j}{r_{ij}}-\frac{\partial}{\partial\vec{r}_i}\left(\vec{p}_j\cdot\frac{\partial r_{ij}}{\partial\vec{r}_i}\right)\right]=2\left[\frac{\vec{p}_j}{r_{ij}}-\frac{1}{2}\frac{\partial}{\partial\vec{r}_i}\left(\vec{p}_j\cdot\frac{\partial}{\partial\vec{r}_i}r_{ij}\right)\right] \;\|$$

Lemma 2°

$$\vec{\nabla}_i V_{ij}=2\vec{\zeta},\qquad V_{ij}=\frac{\vec{p}_i\cdot\vec{p}_j}{r_{ij}}+\frac{\vec{p}_i\cdot\vec{r}_{ij}\,\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^3}$$

$$\vec{\zeta}=\vec{p}_i\times\vec{\nabla}_i\times\left[\frac{\vec{p}_j}{r_{ij}}-\frac{1}{2}\vec{\nabla}_i(\vec{p}_j\cdot\vec{\nabla}_i r_{ij})\right]-(\vec{p}_i\cdot\vec{\nabla}_i)\left[\frac{\vec{p}_j}{r_{ij}}-\frac{1}{2}\vec{\nabla}_i(\vec{p}_j\cdot\vec{\nabla}_i r_{ij})\right]$$

Proof:

$$\frac{\partial}{\partial\vec{r}_i}\left(\vec{p}_j\cdot\frac{\partial}{\partial\vec{r}_i}r_{ij}\right)\equiv\vec{\nabla}_i\left[\vec{p}_j\cdot\vec{\nabla}_i r_{ij}\right]=\vec{\nabla}_i\left[\vec{p}_j\cdot\left(\frac{\vec{r}_{ij}}{r_{ij}}\right)\right]=\vec{\nabla}_i\left[\frac{\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}}\right]$$

$$=\frac{1}{r_{ij}}\vec{\nabla}_i(\vec{p}_j\cdot\vec{r}_{ij})+(\vec{p}_j\cdot\vec{r}_{ij})\vec{\nabla}_i\frac{1}{r_{ij}}$$

$$=\frac{1}{r_{ij}}\vec{p}_j-\frac{\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^3}\vec{r}_{ij},$$

$$\therefore\quad \frac{\vec{p}_j}{r_{ij}}=\vec{\nabla}_i(\vec{p}_j\cdot\vec{\nabla}_i r_{ij})+\frac{\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^3}\vec{r}_{ij}$$

or

$$\frac{\vec{p}_j}{r}=\frac{\partial}{\partial\vec{r}_i}\left(\vec{p}_j\cdot\frac{\partial}{\partial\vec{r}_i}r\right)+\frac{\vec{p}_j\cdot\vec{r}}{r^3}\vec{r},$$

$$\vec{r}\equiv\vec{r}_{ij}=\vec{r}_i-\vec{r}_j$$

$$r\equiv|\vec{r}|\equiv|\vec{r}_{ij}|$$

$$
\begin{aligned}
V_{ij} &= \vec{p}_i \cdot \left(\frac{\vec{p}_j}{r}\right) + \frac{\vec{p}_j \cdot \vec{r}}{r^2}\, \vec{p}_i \cdot \left(\frac{\vec{r}}{r}\right) \\
&= \vec{p}_i \cdot \left(\frac{\vec{p}_j}{r}\right) + \frac{\vec{p}_i \cdot \vec{r}}{r^2}\, \vec{r} \cdot \left(\frac{\vec{p}_j}{r}\right) \\
&= \vec{p}_i \cdot \left[\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} r\right) + \frac{\vec{p}_j \cdot \vec{r}}{r^3}\, \vec{r} \right] \\
&\quad + \frac{\vec{p}_i \cdot \vec{r}}{r^2}\, \vec{r} \cdot \left[\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} r\right) + \frac{\vec{p}_j \cdot \vec{r}}{r^3}\, \vec{r} \right]
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}_i \left(\frac{\vec{p}_i \cdot \vec{p}_j}{r}\right) &= \vec{\nabla}_i \left[\vec{p}_i \cdot \left(\frac{\vec{p}_j}{r}\right) \right] \\
&= \vec{p}_i \times \vec{\nabla}_i \times \left(\frac{\vec{p}_j}{r}\right) + (\vec{p}_i \cdot \vec{\nabla}_i)\left(\frac{\vec{p}_j}{r}\right) + \frac{\vec{p}_j}{r} \times \underbrace{\vec{\nabla}_i \times \vec{p}_i}_{=0} - \left(\frac{\vec{p}_j}{r} \cdot \underbrace{\vec{\nabla}_i\right) \vec{p}_i}_{=0} \\
&= \vec{p}_i \times \frac{\partial}{\partial \vec{r}_i} \times \left(\frac{\vec{p}_j}{r}\right) + \left(\vec{p}_i \cdot \frac{\partial}{\partial \vec{r}_i}\right) \frac{\vec{p}_j}{r}
\end{aligned}
$$

Now consider

$$
\vec{\nabla}_i \left\{ \vec{p}_i \cdot \vec{\nabla}_i \left(\vec{p}_j \cdot \vec{\nabla}_i\, r\right) \right\} = p_i \times \frac{\partial}{\partial \vec{r}_i} \times \left[\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} r\right)\right] + \left(p_i \cdot \frac{\partial}{\partial \vec{r}_i}\right)\left[\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial r}{\partial \vec{r}_i}\right)\right]
$$

$$
+ \left[\vec{\nabla}_i (\vec{p}_j \cdot \vec{\nabla}_i r) \right] \times \left(\frac{\partial}{\partial \vec{r}_i} \times \vec{p}_i \right) + \left[\vec{\nabla}_i (\vec{p}_j \cdot \vec{\nabla}_i r) \cdot \vec{\nabla}_i \right] \vec{p}_i
$$

$$
\therefore \frac{\partial}{\partial \vec{r}_i} \left\{ \vec{p}_i \cdot \frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} |\vec{r}_i - \vec{r}_j|\right) \right\} = \vec{p}_i \times \frac{\partial}{\partial \vec{r}_i} \times \left[\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} |\vec{r}_i - \vec{r}_j|\right) \right] + \left(\vec{p}_i \cdot \frac{\partial}{\partial \vec{r}_i}\right) \left[\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} |\vec{r}_i - \vec{r}_j|\right) \right]
$$

where

$$
r_{ij} \equiv r \equiv \left| \vec{r}_i - \vec{r}_j \right|
$$

We are now ready to prove lemma 2°, thus:

$$
\begin{aligned}
\vec{\zeta} = \vec{p}_i \times \vec{\nabla}_i \times &\left[\frac{\vec{p}_j}{r_{ij}} - \frac{1}{2} \frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} |\vec{r}_i - \vec{r}_j|\right) \right] \\
&+ (\vec{p}_i \cdot \vec{\nabla}_i) \left[\frac{\vec{p}_j}{r_{ij}} - \frac{1}{2} \frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} |\vec{r}_i - \vec{r}_j|\right) \right]
\end{aligned}
$$

$$= \vec{p}_i \times \frac{\partial}{\partial \vec{r}_i} \times \left(\frac{\vec{p}_j}{r_{ij}}\right) + \left(\vec{p}_i \cdot \frac{\partial}{\partial \vec{r}_i}\right)\left(\frac{\vec{p}_j}{r_{ij}}\right)$$

$$- \frac{1}{2}\left\{ \vec{p}_i \times \frac{\partial}{\partial \vec{r}_i} \times \frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} r_{ij}\right) + \left(\vec{p}_i \cdot \frac{\partial}{\partial \vec{r}_i}\right)\left[\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i} r_{ij}\right)\right]\right\}$$

$$= \frac{\partial}{\partial \vec{r}_i}\left(\frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}}\right) - \frac{1}{2}\left\{ \frac{\partial}{\partial \vec{r}_i}\left[\vec{p}_i \cdot \frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i}|\vec{r}_i - \vec{r}_j|\right)\right]\right\}$$

$$= \frac{\partial}{\partial \vec{r}_i}\left\{\left(\frac{\vec{p}_i \cdot \vec{p}_j}{r}\right) - \frac{1}{2}\left(\vec{p}_i \cdot \frac{\partial}{\partial \vec{r}_i}\right)\left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}_i}|\vec{r}_i - \vec{r}_j|\right)\right\}$$

$$= \vec{\nabla}_i \left\{ \frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}} - \frac{1}{2}\vec{p}_i \cdot \left[\frac{\vec{p}_j}{r_{ij}} - \frac{\vec{p}_j \cdot \vec{r}_{ij}}{r_{ij}^3}\vec{r}_{ij}\right]\right\}$$

$$= \vec{\nabla}_i \left\{ \frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}} - \frac{1}{2}\frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}} + \frac{1}{2}\frac{\vec{p}_j \cdot \vec{r}_{ij}}{r_{ij}^3}\vec{p}_i \cdot \vec{r}_{ij}\right\}$$

$$= \vec{\nabla}_i \left\{ \frac{1}{2}\frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}} + \frac{1}{2}\frac{\vec{p}_i \cdot \vec{r}_{ij}\, \vec{p}_j \cdot \vec{r}_{ij}}{r_{ij}^3}\right\}$$

$$= \frac{1}{2}\vec{\nabla}_i \left\{ \frac{\vec{p}_i \cdot \vec{p}_j}{r_{ij}} + \frac{\vec{p}_i \cdot \vec{r}_{ij}\, \vec{p}_j \cdot \vec{r}_{ij}}{r_{ij}^3}\right\}$$

$$= \frac{1}{2}\vec{\nabla}_i V_{ij}$$

$$\therefore \vec{\nabla}_i V_{ij} = 2\vec{\zeta} \quad \square$$

We shall write down Hamilton's equations of motion for the classical many-body problem and show the equivalence of Hamilton's equations of motion and the Newton-Lorentz equation of motion.

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V},$$

$$\mathcal{H}_0 = \sum \left\{ \frac{1}{2m}\left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right)^2 + e\varphi(\vec{r}_i, t)\right\}$$

$$\mathcal{V} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}{}' \frac{e^2}{|\vec{r}_i-\vec{r}_j|} - \frac{1}{2}\frac{e^2}{2m^2c^2}\sum_{i=1}^{N}\sum_{j=1}^{N}{}'\left\{\frac{\vec{p}_i\cdot\vec{p}_j}{r_{ij}} + \frac{\vec{p}_i\cdot\vec{r}_{ij}\,\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^3}\right\}$$

$$\frac{\partial\mathcal{V}}{\partial\vec{r}_i} = \vec{\nabla}_i\mathcal{V} = \sum_{j=1}^{N}{}'\vec{\nabla}_i\frac{e^2}{|r_i-r_j|} - \frac{e^2}{2m^2c^2}\sum_{j=1}^{N}{}'\vec{\nabla}_i\mathcal{V}_{ij}$$

$$\frac{\partial\mathcal{V}}{\partial\vec{r}_i} = e\frac{\partial\Phi_i}{\partial\vec{r}_i} - \frac{e}{mc}\left\{\vec{p}_i\times\frac{\partial}{\partial\vec{r}_i}\times\vec{\mathcal{A}}_i + \vec{p}_i\cdot\frac{\partial}{\partial\vec{r}_i}\vec{\mathcal{A}}_i\right\}$$ by lemma 2°,

where $\Phi_i = \sum_{j=1}^{N}{}'\frac{e}{r_{ij}}$, $(\,'\,)$ means $j \neq i$,

$$\vec{\mathcal{A}}_i = \frac{e}{mc}\sum_{j}{}'\left\{\frac{\vec{p}_j}{r_{ij}} - \frac{1}{2}\vec{\nabla}_i(p_j\cdot\vec{\nabla}_i r_{ij})\right\}$$

$$\frac{\partial\mathcal{H}_0}{\partial\vec{r}_i} = e\frac{\partial\varphi_i}{\partial\vec{r}_i} + \frac{1}{m}\left\{(\vec{p}_i-\tfrac{e}{c}\vec{A}_i)\times\vec{\nabla}_i\times(\vec{p}_i-\tfrac{e}{c}\vec{A}_i) + (\vec{p}_i-\tfrac{e}{c}\vec{A}_i)\vec{\nabla}_i(\vec{p}_i-\tfrac{e}{c}\vec{A}_i)\right\}$$

$$= e\vec{\nabla}_i\varphi_i + \frac{1}{m}\left\{\vec{P}_i\times\left(-\tfrac{e}{c}\vec{\nabla}_i\times\vec{A}_i\right) + (\vec{P}_i\cdot\vec{\nabla}_i)(-\tfrac{e}{c}\vec{A}_i)\right\}$$

$$= e\vec{\nabla}_i\varphi_i - \frac{e}{mc}\left\{\vec{P}_i\times\vec{\nabla}_i\times\vec{A}_i + \vec{P}_i\cdot\vec{\nabla}_i\vec{A}_i\right\}$$

$$\vec{P}_i = \vec{p}_i - \frac{e}{c}\vec{A}_i$$

$$\frac{\partial\mathcal{H}}{\partial\vec{r}_i} = \frac{\partial\mathcal{H}_0}{\partial\vec{r}_i} + \frac{\partial\mathcal{V}}{\partial\vec{r}_i}$$

$$= e\frac{\partial\varphi_i}{\partial\vec{r}_i} + \frac{1}{m}\left\{\vec{P}_i\times\vec{\nabla}_i\times\vec{P}_i + (\vec{P}_i\cdot\vec{\nabla})\vec{P}_i\right\}$$

$$+ e\frac{\partial\Phi_i}{\partial\vec{r}_i} - \frac{1}{m}\left\{\vec{p}_i\times\vec{\nabla}_i\times\vec{\mathcal{A}}_i + \vec{p}_i\cdot\vec{\nabla}_i\vec{\mathcal{A}}_i\right\}$$

$$\frac{\partial \mathcal{V}}{\partial \vec{p}_i} = -\frac{e^2}{2m^2c^2}\sum_{j=1}^{N}{}' \left\{ \frac{\vec{p}_j}{|\vec{r}_i-\vec{r}_j|} + \frac{\vec{p}_j\cdot\vec{r}_{ij}\,\vec{r}_{ij}}{r_{ij}^3} \right\}$$

$$= -\frac{e^2}{2m^2c^2}\, 2\sum_{j=1}^{N}{}' \left\{ \frac{\vec{p}_j}{r_{ij}} - \frac{1}{2}\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_j\cdot\frac{\partial}{\partial \vec{r}_i} r_{ij}\right)\right\}$$

$$= -\frac{e}{mc}\vec{\mathcal{A}}_i$$

$$\frac{\partial \mathcal{H}_0}{\partial \vec{p}_i} = \frac{1}{m}\left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right)$$

$$\vec{v}_i \equiv \frac{d\vec{r}_i}{dt} = \frac{\partial \mathcal{H}}{\partial \vec{p}_i} = \frac{\partial \mathcal{H}_0}{\partial \vec{p}_i} + \frac{\partial \mathcal{V}}{\partial \vec{p}_i} = \frac{1}{m}\left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right) - \frac{e}{mc}\vec{\mathcal{A}}_i$$

$$\vec{p}_i = m\vec{v}_i + \frac{e}{c}\left(\vec{A}_i + \vec{\mathcal{A}}_i\right)$$

where

$\vec{p}_i =$ generalized total momentum of ith particle

$\vec{A}_i =$ vector potential of the externally applied field

$\varphi_i =$ scalar potential of the externally applied field

$\vec{\mathcal{A}}_i =$ vector potential of internal field produced by all of the particles except particle i

$\Phi_i =$ scalar potential of internal field resulting from all of the particles in the plasma except the ith particle.

$$\frac{d\vec{p}_i}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{r}_i} = -e\frac{\partial}{\partial \vec{r}_i}\left(\varphi_i + \Phi_i\right) + \frac{1}{m}\Bigg\{\Big[\left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right)\times\frac{\partial}{\partial \vec{r}_i}\times\left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right)$$

$$+\left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right)\cdot\frac{\partial}{\partial \vec{r}_i}\left(\vec{p}_i - \frac{e}{c}\vec{A}_i\right)\Big] - \left[\vec{p}_i\times\frac{\partial}{\partial \vec{r}_i}\times\vec{\mathcal{A}}_i + \vec{p}_i\cdot\frac{\partial}{\partial \vec{r}_i}\vec{\mathcal{A}}_i\right]\Bigg\}$$

$$\frac{d\vec{p}_i}{dt} = \frac{d}{dt}\left[m\vec{v}_i + \frac{e}{c}(\vec{A}_i + \vec{\mathcal{A}}_i)\right]$$

$$m\frac{d\vec{v}_i}{dt} + \frac{e}{c}\frac{d}{dt}(\vec{A}_i + \vec{\mathcal{A}}_i) = -e\vec{\nabla}_i(\varphi_i + \Phi_i) + \frac{1}{m}\left\{\left[(m\vec{v}_i + \frac{e}{c}\vec{\mathcal{A}}_i)\times\frac{\partial}{\partial\vec{r}_i}(-\frac{e}{c}\vec{A}_i)\right.\right.$$

$$\left. + (m\vec{v}_i + \frac{e}{c}\vec{\mathcal{A}}_i)\cdot\frac{\partial}{\partial\vec{r}_i}(-\frac{e}{c}\vec{A}_i)\right] - \left[\left[m\vec{v}_i + \frac{e}{c}(\vec{A}_i + \vec{\mathcal{A}}_i)\right]\times\frac{\partial}{\partial\vec{r}_i}\times\vec{\mathcal{A}}_i + \right.$$

$$\left.\left.\left[m\vec{v}_i + \frac{e}{c}(\vec{A}_i + \vec{\mathcal{A}}_i)\right]\cdot\frac{\partial}{\partial\vec{r}_i}\vec{\mathcal{A}}_i\right]\right\},$$

$$m\frac{d\vec{v}_i}{dt} + \frac{e}{c}\frac{d}{dt}(\vec{A}_i + \vec{\mathcal{A}}_i) = -e\vec{\nabla}_i(\varphi_i + \Phi_i) - \frac{e}{mc}\left\{(m\vec{v}_i + \frac{e}{c}\vec{\mathcal{A}}_i)\times\vec{B}_i\right.$$

$$\left. + (m\vec{v}_i + \frac{e}{c}\vec{\mathcal{A}}_i)\cdot\vec{\nabla}_i A_i\right\} - \frac{e}{mc}\vec{\omega}$$

where $$\vec{\omega} = m\vec{v}_i \times \vec{\mathcal{B}}_i + \frac{e}{c}(\vec{A}_i + \vec{\mathcal{A}}_i)\times\vec{\mathcal{B}}_i + m\vec{v}_i\cdot\vec{\nabla}_i\vec{\mathcal{A}}$$

$$+ \frac{e}{c}(\vec{A}_i + \vec{\mathcal{A}}_i)\cdot\vec{\nabla}_i\vec{\mathcal{A}}_i$$

$$\vec{\mathcal{B}}_i = \frac{\partial}{\partial\vec{r}_i}\times\vec{\mathcal{A}}_i, \qquad \vec{B}_i = \frac{\partial}{\partial\vec{r}_i}\times\vec{A}_i$$

$$\vec{\mathcal{A}}_i = \frac{e}{mc}\sum_j{}'\left\{\frac{\vec{p}_j}{|\vec{r}_i - \vec{r}_j|} - \frac{1}{2}\frac{\partial}{\partial\vec{r}_i}\left[\vec{p}_j\cdot\frac{\partial}{\partial\vec{r}_i}|\vec{r}_i - \vec{r}_j|\right]\right\}$$

$$\vec{E}_i \equiv \vec{E}(\vec{r}_i,t) = -\frac{\partial\varphi_i}{\partial\vec{r}_i} - \frac{1}{c}\frac{\partial\vec{A}_i}{\partial t} = -\vec{\nabla}_i\varphi(\vec{r}_i,t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{r}_i,t)$$

$$\vec{\mathcal{E}}_i \equiv \vec{\mathcal{E}}(\vec{r}_i,t) = -\vec{\nabla}_i\Phi(\vec{r}_i,t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{\mathcal{A}}(\vec{r}_i,t)$$

$$\Phi_i \equiv \Phi(\vec{r}_i,t) = \sum_{j=1}^{N}{}'\frac{e}{|\vec{r}_i - \vec{r}_j|}$$

$$m\frac{d\vec{v}_i}{dt} + \frac{e}{c}\left\{ \frac{d}{dt}(\vec{A}_i + \vec{\mathcal{A}}_i) - \frac{\partial}{\partial t}(\vec{A}_i + \vec{\mathcal{A}}_i) - (\vec{v}_i\cdot\vec{\nabla}_i)(\vec{A}_i + \vec{\mathcal{A}}_i)\right\}$$

$$= e(\vec{E}_i + \vec{\mathcal{E}}_i) + \frac{e}{c}\vec{v}_i \times (\vec{B}_i + \vec{\mathcal{B}}_i) + \frac{e^2}{m^2c^2}\left\{(\vec{A}_i + \vec{\mathcal{A}}_i)\times\vec{\mathcal{B}}_i\right.$$

$$\left. + (A_i + \mathcal{A}_i)\cdot\vec{\nabla}_i\vec{\mathcal{A}}_i + \vec{\mathcal{A}}_i \times \vec{B}_i + \vec{\mathcal{A}}_i\cdot\vec{\nabla}_i\vec{A}_i\right\} \qquad A-1$$

because

$$\vec{\mathcal{A}}_i = \frac{e}{mc}\sum_{j=1}^{N}{}' \left[\frac{\vec{p}_j}{|\vec{r}_i - \vec{r}_j|} - \frac{1}{2}\frac{\partial}{\partial\vec{r}_i}\left(\vec{p}_j\cdot\frac{\partial}{\partial\vec{r}_i}|\vec{r}_i - \vec{r}_j|\right)\right], \quad \vec{\mathcal{B}}_i = \vec{\nabla}_i \times \vec{\mathcal{A}}_i .$$

The last term on the right hand side of the above equation is of order $\frac{v^3}{c^3}$, i.e.

$$m\frac{d\vec{v}_i}{dt} = e(\vec{E}_i + \vec{\mathcal{E}}_i) + \frac{e}{c}\vec{v}_i \times (\vec{B}_i + \vec{\mathcal{B}}_i) + O\left(\frac{v^3}{c^3}\right),$$

because the second term on the left hand side of equation A-1 vanishes identically. So

$$m\frac{d\vec{v}_i}{dt} \cong e(\vec{\mathcal{E}}_i + \vec{E}_i) + \frac{e}{c}\vec{v}_i \times (\vec{B}_i + \vec{\mathcal{B}}_i)$$

where $\vec{E}_i = \vec{E}(\vec{r}_i, t)$, $\vec{B}_i = \vec{B}(\vec{r}_i, t)$ are the given external fields, and $\vec{\mathcal{E}}(\vec{r}_i, t)$ and $\vec{\mathcal{B}}(\vec{r}_i, t)$ the fields produced by all particles (except the ith), that is, $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are the solutions of Maxwell's equations with charge and current densities given by

$$\rho(\vec{r}, t) = e\sum_{j=1}^{N}{}'\delta(\vec{r} - \vec{r}_j), \quad \vec{J}(\vec{r}, t) = e\sum_{j=1}^{N}{}'\vec{v}_j\,\delta(\vec{r} - \vec{r}_j),$$

and

$$\vec{\mathcal{B}} = \vec{\nabla} \times \vec{\mathcal{A}}, \quad \vec{\mathcal{A}} = \frac{e}{mc} \sum_{j=1}^{N}{}' \left[\frac{\vec{p}_j}{|\vec{r}-\vec{r}_j|} - \frac{1}{2} \frac{\partial}{\partial \vec{r}} \left(\vec{p}_j \cdot \frac{\partial}{\partial \vec{r}} |\vec{r}-\vec{r}_j| \right) \right],$$

where the subscript "i" for $\vec{\mathcal{E}}_i$, $\vec{\mathcal{B}}_i$, $\vec{\mathcal{A}}_i$ etc. has been consistently dropped.

III-5 Liouville's Theorem for a Many-Particle System and Its Reduction

We consider in this section the integration of Liouville's equation for a many-particle system in Γ-space.

The fact that Liouville's distribution function $F^{(n)}$ is a function of the coordinates and momenta of all the particles as well as the time causes not only mathematical complexity but conceptual difficulty as well. Conceptual difficulty arises primarily because normally we do not think in terms of distributions in large numbers of particles but rather in terms of distribution functions that are based on one or possibly two or three particles.

Moreover, the many-particle distribution function contains a great deal of information including, for example, interparticle correlations of various orders. For most physically interesting situations one anticipates only a very small number of these correlations to play an important role. Thus, it is clear that $F^{(n)}$ should hopefully be reducable to distribution functions involving only small numbers of particles by averaging out the coordinates and momenta of the remaining large collection of particles.

For simplicity, we consider a system of charged particles of only one species and that $F^{(n)}$ is symmetric with respect to exchange of the coordinates and momenta of any two particles. The S-particle distribution function $F^{(S)}$ $(S \ll N)$ is obtained by averaging over the coordinates and momenta of the remaining particles, i.e.

$$F^{(S)}(t) = \int F^{(N)}(t) \prod_{j=S+1}^{N} d^6 \zeta_j$$

where

$$F^{(N)}(t) \equiv F^{(N)}(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N, \vec{p}_1, \vec{p}_2, \cdots \vec{p}_N, t)$$

$$\prod_{j=S+1}^{N} d^6 \zeta_j = \prod_{j=S+1}^{N} d^3 r_j \, d^3 p_j = (d^3 r_{S+1} d^3 p_{S+1})(d^3 r_{S+2} d^3 p_{S+2}) \cdots (d^3 r_N d^3 p_N),$$

and has the interpretation that the quantity

$$\prod_{i=1}^{S} d^6 \zeta_i \, F^{(S)}(t) = d\Omega \, F^{(S)}(t)$$

is the joint probability at time t that a particular collection of S particles will be located (in physical space) at positions $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_S$ in the ranges $d^3 r_1, d^3 r_2, \cdots, d^3 r_S$ respectively, and with momenta $\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_S$ in the ranges $d^3 p_1, d^3 p_2, \cdots d^3 p_S$, independent of the locations and the states of motion of the remaining $N-S$ particles.

We have shown in Section II-1 that Liouville's theorem for a many-particle system governed by a Hamiltonian H can be written as

$$\frac{dF^{(N)}}{dt} + \left[F^{(N)}, \mathcal{H} \right] = 0$$

or more explicitly,

$$\frac{dF^{(N)}}{dt} + \sum_{i=1}^{N}\left\{ \frac{\partial F^{(N)}}{\partial \vec{r}_i}\frac{\partial \mathcal{H}}{\partial \vec{p}_i} - \frac{\partial F^{(N)}}{\partial \vec{p}_i}\frac{\partial \mathcal{H}}{\partial \vec{r}_i}\right\} = 0 \qquad 1$$

We now integrate (1) over the coordinates and momenta of all the particles except, say, for those particles numbered 1,2,3,...S, where $S \ll N$, thus:

$$\int \frac{dF^{(N)}}{dt}\prod_{j=S+1}^{N} d^6\zeta_j + \sum_{i=1}^{S}\int\left\{ \frac{\partial F^{(N)}}{\partial \vec{r}_i}\frac{\partial \mathcal{H}}{\partial \vec{p}_i} - \frac{\partial F^{(N)}}{\partial \vec{p}_i}\frac{\partial \mathcal{H}}{\partial \vec{r}_i}\right\}\prod_{j=S+1}^{N} d^6\zeta_j$$
$$+\sum_{i=S+1}^{N}\int\left\{ \frac{\partial F^{(N)}}{\partial \vec{r}_i}\frac{\partial \mathcal{H}}{\partial \vec{p}_i} - \frac{\partial F^{(N)}}{\partial \vec{p}_i}\frac{\partial \mathcal{H}}{\partial \vec{r}_i}\right\}\prod_{j=S+1}^{N} d^6\zeta_j = 0,$$

$$i.e.\ \frac{\partial}{\partial t}\int F^{(N)} d\Omega + \sum_{i=S+1}^{N}\left\{ \frac{\partial \mathcal{H}}{\partial \vec{p}_i}\frac{\partial}{\partial \vec{r}_i} - \frac{\partial \mathcal{H}}{\partial \vec{r}_i}\frac{\partial}{\partial \vec{p}_i}\right\}\int F^{(N)} d\Omega +$$

$$\sum\int\left\{\frac{\partial}{\partial \vec{r}_i}\cdot\left[F^{(N)}\frac{\partial \mathcal{H}}{\partial \vec{p}_i}\right] - \frac{\partial}{\partial \vec{p}_i}\cdot\left[F^{(N)}\frac{\partial \mathcal{H}}{\partial \vec{r}_i}\right] + F^{(N)}\left[\frac{\partial}{\partial \vec{p}_i}\cdot\frac{\partial \mathcal{H}}{\partial \vec{r}_i} - \frac{\partial}{\partial \vec{r}_i}\cdot\frac{\partial \mathcal{H}}{\partial \vec{p}_i}\right]\right\}\prod_{j=S+1}^{N} d^6\zeta_j = 0$$

$$\therefore\ \frac{\partial}{\partial t}F^{(S)} + \sum_{i=1}^{S}\left\{ \frac{\partial F^{(S)}}{\partial \vec{r}_i}\frac{\partial \mathcal{H}}{\partial \vec{p}_i} - \frac{\partial F^{(S)}}{\partial \vec{p}_i}\frac{\partial \mathcal{H}}{\partial \vec{r}_i}\right\} + \sum_{i=S+1}^{N}\int\left\{\frac{\partial}{\partial \vec{r}_i}\cdot\left[F^{(N)}\dot{\vec{r}}_i\right] + \frac{\partial}{\partial \vec{p}_i}\left[F^{(N)}\dot{\vec{p}}_i\right]\right\}\prod_{j=S+1}^{N} d^6\zeta_j = 0$$

The last term can be evaluated in μ-space by Gauss's theorem as follows:

$$\sum_{i=S+1}^{N}\int\left\{\frac{\partial}{\partial \zeta_i}\cdot\left[F^{(N)}\dot{\zeta}_i\right]\right\}\prod_{j=S+1}^{N} d^6\zeta_j = \sum_{i=S+1}^{N}\oint\prod_{j=S+1}^{N} F^{(N)}\dot{\zeta}_i\cdot d^6S_j = 0$$

where

$$\frac{\partial}{\partial \zeta_i}\cdot\left[F^{(N)} \dot{\zeta}_i \right] = \frac{\partial}{\partial \vec{r}_i}\cdot\left[F^{(N)} \dot{\vec{r}}_i \right] + \frac{\partial}{\partial \vec{p}_i}\cdot\left[F^{(N)} \dot{\vec{p}}_i \right]$$

and

$$\frac{\partial}{\partial \zeta_i}\cdot \longrightarrow \frac{\partial}{\partial \vec{r}_i}\cdot \quad + \frac{\partial}{\partial \vec{p}_i}\cdot \longrightarrow \vec{\nabla}_i\cdot \; + \vec{\nabla}_{\vec{p}_i}\cdot$$

Consequently,

$$\frac{\partial F^{(S)}}{\partial t} + \left[F^{(S)}, \mathcal{H} \right] = 0$$

To proceed further, we separate the Hamiltonian into two parts, namely, a non-interacting part H_o and an interacting part V, thus:

$$\mathcal{H} = \mathcal{H}_o + V,$$

and Liouville's equation becomes

$$\frac{\partial F^{(S)}}{\partial t} + \left[F^{(S)}, \mathcal{H}_o \right] = \left[V, F^{(S)} \right]$$

or

$$\frac{\partial F^{(S)}}{\partial t} + \sum_{i=1}^{S} \left\{ \frac{\partial F^{(S)}}{\partial \vec{r}_i}\cdot\frac{\partial \mathcal{H}_o}{\partial \vec{p}_i} - \frac{\partial F^{(S)}}{\partial \vec{p}_i}\cdot\frac{\partial \mathcal{H}_o}{\partial \vec{r}_i} \right\} = \sum_{i=1}^{S} \left\{ \frac{\partial V}{\partial \vec{r}_i}\cdot\frac{\partial F^{(S)}}{\partial \vec{p}_i} - \frac{\partial V}{\partial \vec{p}_i}\cdot\frac{\partial F^{(S)}}{\partial \vec{r}_i} \right\}.$$

We shall use this convenient S-particle distribution function to derive the famous Bogoliubov-Born-Green-Kirkwood-Yvon Hierarchy for a fully ionized plasma with Coulomb interaction in the relativistic form as well as in the non-relativistic form in the following section.

CHAPTER FOUR

THE BOLTZMANN-VLASOV EQUATION ON THE BASIS OF THE SELF-CONSISTENT FIELDS

IV-1 Introduction

We shall derive the Vlasov-Boltzmann equations on the basis of the self-consistent field approximation. In order to gain a deeper insight into the physics involved, let us consider first the related problem of obtaining a mathematically tractable equation from Liouville's theorem for the case of a dilute neutral gas. This system is basically different from a plasma in that the intermolecular forces are short range in contrast to the long-range Coulomb forces in a plasma. For the case of a dilute neutral gas, Bogoliubov has given a new derivation of the Boltzmann equation by expanding the solution of Liouville's equation in a power series in terms of a small parameter α which is given by

$$\alpha = n\, r_0^3$$

where n is the particle density, and r_0 is the range of molecular force. For the case of a gas consisting exclusively of neutral particles, the force range r_0 is of order 10^{-8}cm and thus even for densities as large as 10^{20} particles per cubic centimeter α is negligible compared to unity. The magnitude of the expansion parameter α may be thought of as being a measure of the probability that various numbers of particles will collide simultaneously. Thus, if α is very small compared to unity, then it is very unlikely that more than two particles will ever suffer a simultaneous collision; and, consequently, one expects that the Boltzmann equation, which allows only for binary collisions, will correctly describe the system. On the other hand, for liquids where α is of order unity, one expects that the Boltzmann equation can be modified such that three or four particle-collision terms are considered. Just like the three body problem in classical mechanics, these multi-particle "collisions" complicate the analysis enormously and are one of the reasons why a satisfactory theory of the liquid state is still lacking.

Consider next the possibility of using the binary-collision form of the Boltzmann equation to describe a plasma. It is easy to see that α will always be very large regardless of the particle density because the range of the Coulomb force is essentially infinite, and that Bogoliubov's approach would lead to a divergent power series. In terms of our interpretation of the parameter α, this would mean that most of the particles in the plasma are constantly in simultaneous collision. However, the physical picture is not quite so because each particle in the plasma is screened from other particles which are outside the Debye sphere with shielding radius

$$\delta = \left\{ \frac{kT}{4\pi n e^2} \right\}^{1/2}$$

where n is the particle density, e is the electronic charge, and T represents the absolute temperature of the plasma. It is the Debye radius that should be used as the range, in the expansion parameter, thus:

$$\alpha = n\delta^3 = \left[\frac{k}{4\pi e^2}\right]^{3/2} T^{3/2}\, n^{-1/2}$$
$$= \left[\frac{k}{4\pi e^2}\right]^{3/2} \sqrt{\frac{T^3}{n}}$$

and this shows that on this basis as well, α is still very large compared to unity for most physically interesting plasmas. For example, at a temperature of 10^5K, as the particle density varies between 10^{16} and 10^{12} α covers the range from 10 to 1000; consequently, the particles in the plasma will suffer repeated, multiparticle, and simultaneous collisions and so the binary-collision form of the Boltzmann equation is no longer valid for a plasma. This uncomfortable situation might appear frustrating; however, we shall demonstrate that this simultaneous multiparticle collision is actually a saving grace for simplifying the theoretical treatment of a plasma because this is the very basis on which the self-consistent field method is built. Qualitatively, it is easy to see for a large value of α, that only if the particles move in a coherent or collective fashion will a typical particle in the plasma experience an appreciable force. Furthermore, such forces can be expected to be much more smoothly behaved functions of space and time than the forces produced by the much rarer multi-particle "collisions". Thus, to lowest order we could replace all interparticle forces by microscopically smoothed electromagnetic fields. Each particle moves independently in the microscopic smoothed electromagnetic fields produced by all the other particles, and each particle has a well defined trajectory just like the independent particle model of the nucleus in which each nucleon moves independently with a well defined orbit in the average potential produced by all the other nucleons. The independent trajectories of the particles give rise to microscopic charge and current densities which in turn produce those fields that will determine the motions of the particles in a self-consistent way. It will be shown in the next section that this method of solution turns out to be equivalent to the one obtained by solving for the single-particle distribution function (from that form) of the Boltzmann equation that is obtained by omitting the collision term but supplementing the external force terms in this equation with the smoothed, self-consistent fields. The fields are obtained by solving Maxwell's equations with charge and current density expressed in terms of the distribution function, and this set of equations, namely, the collisionless Boltzmann equation plus the above described form of Maxwell equations, will be referred to as the Boltzmann-Vlasov equation from now on.

IV-2 The Boltzmann-Vlasov Equation

The N-particle Liouville equation is

$$\frac{\partial F^{(N)}}{\partial t} + \sum_{i=1}^{N} \left\{ \frac{\partial F^{(N)}}{\partial \vec{r}_i} \cdot \frac{\partial \mathcal{H}}{\partial \vec{p}_i} - \frac{\partial F^{(N)}}{\partial \vec{p}_i} \cdot \frac{\partial \mathcal{H}}{\partial \vec{r}_i} \right\} = 0$$

or

$$\frac{\partial F^{(N)}}{\partial t} + \sum_{i=1}^{N} \left\{ \frac{\partial F^{(N)}}{\partial \vec{r}_i} \cdot \frac{\partial \mathcal{H}_0}{\partial \vec{p}_i} - \frac{\partial F^{(N)}}{\partial \vec{p}_i} \cdot \frac{\partial \mathcal{H}_0}{\partial \vec{r}_i} \right\} = \sum_{i=1}^{N} \left\{ \frac{\partial F^{(N)}}{\partial \vec{p}_i} \frac{\partial V}{\partial \vec{r}_i} - \frac{\partial F^{(N)}}{\partial \vec{r}_i} \frac{\partial V}{\partial \vec{p}_i} \right\} \qquad 1$$

where $\mathcal{H} = \mathcal{H}_0 + V$ is the Darwin Hamiltonian, i.e.

$$\mathcal{H}_0 = \sum_{i=1}^{N} \frac{[\vec{p}_i - \frac{e}{c}\vec{A}_i]^2}{2m} + \sum_{i=1}^{N} e\varphi(\vec{r}_i\, t),$$

$$V = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}{}' \frac{e^2}{|\vec{r}_i - \vec{r}_j|} - \frac{1}{2}\frac{e^2}{2m^2c^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\{\frac{\vec{p}_i\cdot\vec{p}_j}{|r_i - r_j|} + \frac{\vec{p}_i\cdot(\vec{r}_i-\vec{r}_j)\,\vec{p}_j\cdot(\vec{r}_i-\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|^3}\right\}.$$

If equation (1) is integrated over the coordinates and momenta of all of the particles except those for, say number 1, we find

$$\frac{\partial F^{(1)}}{\partial t} + \frac{\partial \mathcal{H}_0}{\partial \vec{p}_1}\frac{\partial F^{(1)}}{\partial \vec{r}_1} - \frac{\partial \mathcal{H}_0}{\partial \vec{r}_1}\frac{\partial F^{(1)}}{\partial \vec{p}_1} = \int \left\{\frac{\partial V}{\partial \vec{r}_1}\frac{\partial F^{(N)}}{\partial \vec{p}_1} - \frac{\partial F^{(N)}}{\partial \vec{r}_1}\frac{\partial V}{\partial \vec{p}_1}\right\} \prod_{i=2}^{N} d^6\zeta_i,$$

$$\frac{\partial F^{(1)}}{\partial t} + \frac{\partial \mathcal{H}_0}{\partial \vec{p}_1}\frac{\partial F^{(1)}}{\partial \vec{r}_1} - \frac{\partial \mathcal{H}_0}{\partial \vec{r}_1}\frac{\partial F^{(1)}}{\partial \vec{p}_1} = \int d^6\zeta_2 \left\{\frac{\partial V}{\partial \vec{r}_1}\frac{\partial F^{(2)}}{\partial \vec{p}_1} - \frac{\partial V}{\partial \vec{p}_1}\frac{\partial F^{(2)}}{\partial \vec{r}_1}\right\}$$

where

$$F^{(2)} = \int F^{(N)} \prod_{i=3}^{N} d^6\zeta_i = \int \prod_{j=1}^{N} f_j \prod_{i=3}^{N} d^6\zeta_i$$

$$= \int f_1 f_2 \prod_{j=3}^{N} f_j \prod_{i=3}^{N} d^6\zeta_i = f_1 f_2 \prod_{j=3}^{N} \int f_j\, d^6\zeta_j$$

$$= f(\vec{r}_1, \vec{p}_1, t)\, f(\vec{r}_2, \vec{p}_2, t)$$

and where the second step

$$F^{(N)} = \prod_{j=1}^{N} f_j \quad , \qquad f_j \equiv f(\vec{r}_j, \vec{p}_j, t)$$

follows from the independent trajectory Ansatz. Similarly

$$\begin{aligned} F^{(1)} &= \int F^{(N)} \prod_{i=2}^{N} d^6\zeta_i = \int \prod_{j=1}^{N} f_j \prod_{i=2}^{N} d^6\zeta_i \\ &= f_1 \int \prod_{j=2}^{N} f_j \prod_{i=2}^{N} d^6\zeta_i = f_1 \prod_{j=2}^{N} \int f_j \, d^6\zeta_j \\ &= f(\vec{r}_1, \vec{p}_1, t) \end{aligned}$$

and

$$\int f_j \, d^6\zeta_j = 1 \, .$$

To proceed further, we state the following lemmas (these were proven before):

Lemma 1°

$$\frac{\partial}{\partial \vec{p}_1} \left[\frac{\vec{p}_1 \cdot \vec{p}_2}{r_{12}} + \frac{\vec{p}_1 \cdot \vec{r}_{12} \, \vec{p}_2 \cdot \vec{r}_{12}}{r_{12}^3} \right] = 2 \left[\frac{\vec{p}_2}{r_{12}} - \frac{1}{2} \frac{\partial}{\partial \vec{r}_1} \left(\vec{p}_2 \cdot \frac{\partial}{\partial \vec{r}_1} r_{12} \right) \right]$$

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 \quad , \qquad r_{12} = \left| \vec{r}_1 - \vec{r}_2 \right|$$

Lemma 2°

$$\frac{\partial}{\partial \vec{r}_1}\left[\frac{\vec{p}_1\cdot\vec{p}_2}{r_{12}}+\frac{\vec{p}_1\cdot\vec{r}_{12}\,\vec{p}_2\cdot\vec{r}_{12}}{r_{12}^3}\right]=2\vec{\xi}$$

where

$$\vec{\xi}=\vec{p}_1\times\left\{\frac{\partial}{\partial\vec{r}_1}\times\left[\frac{\vec{p}_2}{r_{12}}-\frac{1}{2}\frac{\partial}{\partial\vec{r}_1}\left(\vec{p}_2\cdot\frac{\partial}{\partial\vec{r}_1}r_{12}\right)\right]\right\}$$

$$+\left(\vec{p}_1\cdot\frac{\partial}{\partial\vec{r}_1}\right)\left[\frac{\vec{p}_2}{r_{12}}-\frac{1}{2}\frac{\partial}{\partial\vec{r}_1}\left(\vec{p}_2\cdot\frac{\partial}{\partial\vec{r}_1}r_{12}\right)\right]$$

Lemma 3°

$$2\vec{\xi}=\frac{1}{r_{12}^3}\left\{\left[(\vec{p}_1\cdot\vec{r}_{12})\vec{p}_2+(\vec{p}_2\cdot\vec{r}_{12})\vec{p}_1\right]\right.$$
$$\left.-\left[\vec{p}_1\cdot\vec{p}_2-\frac{3(\vec{p}_1\cdot\vec{r}_{12})(\vec{p}_2\cdot\vec{r}_{12})}{r_{12}^2}\right]\vec{r}_{12}\right\}$$

$$\frac{\partial \mathcal{H}_0}{\partial\vec{p}_1}=\frac{1}{m}\left(\vec{p}_1-\frac{e}{c}\vec{A}_1\right),$$

$$\frac{\partial \mathcal{H}_0}{\partial\vec{r}_1}=\frac{1}{m}\left\{\left(\vec{p}_1-\frac{e}{c}\vec{A}_1\right)\times\frac{\partial}{\partial\vec{r}_1}\times\left(\vec{p}_1-\frac{e}{c}\vec{A}_1\right)+\left(\vec{p}_1-\frac{e}{c}\vec{A}_1\right)\cdot\frac{\partial}{\partial\vec{r}_1}\left(\vec{p}_1-\frac{e}{c}\vec{A}_1\right)\right\}+e\frac{\partial\varphi_1}{\partial\vec{r}_1}$$

$$\frac{\partial V}{\partial\vec{p}_1}=-\frac{e^2}{2m^2c^2}\sum_{j=2}^{N}{}'\left\{\frac{\vec{p}_j}{r_{1j}}+\frac{\vec{p}_j\cdot\vec{r}_{1j}}{r_{1j}^3}\vec{r}_{1j}\right\}\qquad r_{1j}=\left|\vec{r}_1-\vec{r}_j\right|$$

$$\frac{\partial V}{\partial\vec{r}_1}=\frac{\partial}{\partial\vec{r}_1}\sum_{j=2}^{N}{}'\frac{e^2}{|\vec{r}_1-\vec{r}_j|}-\frac{e^2}{2m^2c^2}\sum_{j=2}^{N}{}'\frac{\partial}{\partial\vec{r}_1}\left\{\frac{\vec{p}_1\cdot\vec{p}_j}{r_{1j}}+\frac{(\vec{p}_1\cdot\vec{r}_{1j})(\vec{p}_j\cdot\vec{r}_{1j})}{r_{1j}^3}\right\}$$

$$\frac{\partial F^{(1)}}{\partial t}+\frac{1}{m}(\vec{p}_1-\frac{e}{c}\vec{A}_1)\cdot\frac{\partial F^{(1)}}{\partial \vec{r}_1}-\frac{1}{m}\Big\{(\vec{p}_1-\frac{e}{c}\vec{A}_1)\times\frac{\partial}{\partial \vec{r}_1}\times(\vec{p}_1-\frac{e}{c}\vec{A}_1)+$$

$$(\vec{p}_1-\frac{e}{c}\vec{A}_1)\cdot\frac{\partial}{\partial \vec{r}_1}(\vec{p}_1-\frac{e}{c}\vec{A}_1)\Big\}\cdot\frac{\partial F^{(1)}}{\partial \vec{p}_1}+e\frac{\partial \varphi_1}{\partial \vec{r}_1}\cdot\frac{\partial F^{(1)}}{\partial \vec{p}_1}=$$

$$=\int d^6\zeta_2\Big\{\Big[\frac{\partial}{\partial \vec{r}_1}\sum_{j=2}^{N}{}'\frac{e^2}{r_{1j}}-\frac{e^2}{2m^2c^2}\sum_{j}^{N}{}'\frac{\partial}{\partial \vec{r}_1}\Big(\frac{\vec{p}_1\cdot\vec{p}_j}{r_{1j}}+\frac{\vec{p}_1\cdot\vec{r}_{1j}\,\vec{p}_j\cdot\vec{r}_{1j}}{r_{1j}^3}\Big)\Big]\frac{\partial F^{(2)}}{\partial \vec{p}_1}$$

$$+\Big[\frac{e^2}{2m^2c^2}\sum_{j}^{N}{}'\frac{\partial}{\partial \vec{p}_1}\Big(\frac{\vec{p}_1\cdot\vec{p}_j}{r_{1j}}+\frac{\vec{p}_1\cdot\vec{r}_{1j}\,\vec{p}_j\cdot\vec{r}_{1j}}{r_{1j}^3}\Big)\Big]\cdot\frac{\partial F^{(2)}}{\partial \vec{p}_1}\Big\}$$

$$=(N-1)\int d^6\zeta_2\Big\{\Big[\frac{\partial}{\partial \vec{r}_1}\Big(\frac{e^2}{r_{12}}\Big)-\frac{e^2}{2m^2c^2}\frac{\partial}{\partial \vec{r}_1}\Big(\frac{\vec{p}_1\cdot\vec{p}_2}{r_{12}}+\frac{\vec{p}_1\cdot\vec{r}_{12}\,\vec{p}_2\cdot\vec{r}_{12}}{r_{12}^3}\Big)\Big]\cdot\frac{\partial F^{(2)}}{\partial \vec{p}_1}$$

$$+\frac{e^2}{2m^2c^2}\Big[\frac{\partial}{\partial \vec{p}_1}\Big(\frac{\vec{p}_1\cdot\vec{p}_2}{r_{12}}+\frac{\vec{p}_1\cdot\vec{r}_{12}\,\vec{p}_2\cdot\vec{r}_{12}}{r_{12}^3}\Big)\Big]\cdot\frac{\partial F^{(2)}}{\partial \vec{r}_1}\Big\}$$

$$\cong N\int d^6\zeta_2\Big\{\Big[\frac{\partial}{\partial \vec{r}_1}\Big(\frac{e^2}{r_{12}}\Big)-\frac{e^2}{2m^2c^2}2\vec{\xi}\Big]\cdot\frac{\partial F^{(2)}}{\partial \vec{p}_1}+\frac{e^2}{2m^2c^2}2\Big[\frac{\vec{p}_2}{r_{12}}-\frac{1}{2}\frac{\partial}{\partial \vec{r}_1}\Big(\vec{p}_2\cdot\frac{\partial}{\partial \vec{r}_1}r_{12}\Big)\Big]\cdot\frac{\partial F^{(2)}}{\partial \vec{r}_1}\Big\}$$

$$\vec{\xi}=\vec{p}_1\times\frac{\partial}{\partial \vec{r}_1}\times\Big[\frac{\vec{p}_2}{r_{12}}-\frac{1}{2}\frac{\partial}{\partial \vec{r}_1}\Big(\vec{p}_2\cdot\frac{\partial}{\partial \vec{r}_1}r_{12}\Big)\Big]+\Big(\vec{p}_1\cdot\frac{\partial}{\partial \vec{r}_1}\Big)\Big[\frac{\vec{p}_2}{r_{12}}-\frac{1}{2}\frac{\partial}{\partial \vec{r}_1}\Big(\vec{p}_2\cdot\frac{\partial}{\partial \vec{r}_1}r_{12}\Big)\Big].$$

By Lemma 2°,

$$\frac{\partial F^{(1)}}{\partial t}+\frac{1}{m}(\vec{p}_1-\frac{e}{c}\vec{A}_1)\frac{\partial F^{(1)}}{\partial \vec{r}_1}+\frac{e}{mc}\Big\{(\vec{p}_1-\frac{e}{c}\vec{A}_1)\times\frac{\partial}{\partial \vec{r}_1}\times\vec{A}_1+(\vec{p}_1-\frac{e}{c}\vec{A}_1)\cdot\frac{\partial}{\partial \vec{r}_1}\vec{A}_1\Big\}\cdot\frac{\partial F^{(1)}}{\partial \vec{p}_1}$$

$$+e\frac{\partial \varphi_1}{\partial \vec{r}_1}\cdot\frac{\partial F^{(1)}}{\partial \vec{p}_1}=\frac{Ne^2}{m}\int d^6\zeta_2\Big[\frac{\partial}{\partial \vec{r}_1}\frac{1}{|\vec{r}_1-\vec{r}_2|}\Big]f_2\cdot\frac{\partial f_1}{\partial \vec{p}_1}-\frac{Ne^2}{m}\int d^6\zeta_2\Big\{\vec{p}_1\times\Big[\frac{\partial}{\partial \vec{r}_1}\times\Big(\frac{\vec{p}_2}{r_{12}}-\frac{1}{2}\frac{\partial}{\partial \vec{r}_1}\Big(\vec{p}_2\cdot\frac{\partial r_{12}}{\partial \vec{r}_1}\Big)\Big)\Big]$$

$$+\vec{p}_1\frac{\partial}{\partial \vec{r}_1}\Big[\frac{\vec{p}_2}{r_{12}}-\frac{1}{2}\frac{\partial}{\partial \vec{r}_1}\Big(\vec{p}_2\cdot\frac{\partial r_{12}}{\partial \vec{r}_1}\Big)\Big]\Big\}\cdot f_2\frac{\partial f_1}{\partial \vec{p}_1}+\frac{Ne^2}{m^2c^2}\int d^6\zeta_2\Big[\frac{\vec{p}_2}{|\vec{r}_1-\vec{r}_2|}-\frac{1}{2}\frac{\partial}{\partial \vec{r}_1}\Big(\vec{p}_2\cdot\frac{\partial}{\partial \vec{r}_1}|\vec{r}_1-\vec{r}_2|\Big)\Big]f_2\cdot\frac{\partial f_1}{\partial \vec{r}_1}.$$

where

$$F^{(2)} = f_1 f_2 = f(\vec{r}_1,\vec{p}_1,t)\, f(\vec{r}_2,\vec{p}_2,t), \quad f_1 \equiv f(\vec{r}_1,\vec{p}_1,t)$$

Define

$$\Phi_1 \equiv \Phi(\vec{r}_1,t) = Ne\int d^3r_2\, d^3p_2\, \frac{1}{|\vec{r}_1-\vec{r}_2|} f(\vec{r}_2,\vec{p}_2,t)$$

$$\vec{\mathcal{A}}_1 \equiv \vec{\mathcal{A}}(\vec{r}_1,t) = \frac{Ne}{mc}\int d^3r_2\, d^3p_2 \left[\frac{\vec{p}_2}{|\vec{r}_1-\vec{r}_2|} - \frac{1}{2}\frac{\partial}{\partial\vec{r}_1}\left(\vec{p}_2\cdot\frac{\partial}{\partial\vec{r}_1}|\vec{r}_1-\vec{r}_2|\right)\right] f(\vec{r}_2,\vec{p}_2,t)$$

Then we have

$$\frac{\partial f_1}{\partial t} + \frac{1}{m}\left[\vec{p}_1 - \frac{e}{c}(\vec{A}_1 + \vec{\mathcal{A}}_1)\right]\cdot\frac{\partial f_1}{\partial\vec{r}_1} + \frac{e}{mc}\left\{\left[\left(p_1 - \frac{e}{c}[\vec{A}_1+\vec{\mathcal{A}}_1]\right)\cdot\frac{\partial}{\partial\vec{r}_1}\right](\vec{A}_1+\vec{\mathcal{A}}_1)\right.$$

$$\left. + \left[p_1 - \frac{e}{c}(\vec{A}_1+\vec{\mathcal{A}}_1)\right]\times(\vec{B}_1+\vec{\mathcal{B}}_1) - mc\vec{\nabla}_1(\varphi_1+\Phi_1)\right\}\cdot\frac{\partial f_1}{\partial\vec{p}_1} = 0$$

$$\vec{\mathcal{B}}_1 = \frac{\partial}{\partial\vec{r}_1}\times\vec{\mathcal{A}}_1\ , \quad \vec{B}_1 = \frac{\partial}{\partial\vec{r}_1}\times\vec{A}_1$$

We now change the variables according to

$$\vec{r}_1,\ \vec{p}_1,\ t \longrightarrow \vec{r}_1,\ \vec{v}_1,\ t\ ,$$

and omit the subscript 1:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial\vec{r}} + \frac{e}{m}\left[\vec{E} + \vec{\mathcal{E}} + \frac{1}{c}\vec{v}\times(\vec{B}+\vec{\mathcal{B}})\right]\cdot\frac{\partial f}{\partial\vec{v}} = 0$$

where all terms of order $\frac{v^3}{c^3}$ have been dropped, and with

$$\vec{\mathcal{E}} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\mathcal{A}}$$

$$\vec{\mathcal{B}} = \vec{\nabla}\times\vec{\mathcal{A}}\ .$$

CHAPTER FIVE

THE BOGOLIULOV-BORN-GREEN-KIRKWOOD-YVON HIERARCHY

V-1 The B-B-G-K-Y Hierarchy in Non-Relativistic Form

We shall first treat the non-relativistic case, and using the Darwin Hamiltonian

$$\mathcal{H} = \sum_i (\vec{P}_i - \tfrac{e}{c}\vec{A}_i)^2/2m + \sum_i e\varphi_i + \frac{1}{2}\sum_i\sum_j{}' \frac{e^2}{|\vec{r}_i - \vec{r}_j|} - \frac{1}{2}\frac{e^2}{2m^2c^2}\sum_i\sum_j{}'\left[\frac{\vec{P}_i\cdot\vec{P}_j}{r_{ij}} + \frac{\vec{P}_i\cdot\vec{r}_{ij}\,\vec{P}_j\cdot\vec{r}_{ij}}{r_{ij}^3}\right]$$

The non-interacting part is seen to be given by

$$\mathcal{H}_o = \sum\left\{\frac{1}{2m}(\vec{P}_i - \tfrac{e}{c}\vec{A}_i)^2 + e\varphi_i\right\}$$

and the interacting part is given by

$$\mathcal{H}_{int} = \frac{1}{2}\sum_i\sum_j{}' \frac{e^2}{|\vec{r}_i - \vec{r}_j|} - \frac{1}{2}\frac{e^2}{2m^2c^2}\sum_i\sum_j{}'\left\{\frac{\vec{P}_i\cdot\vec{P}_j}{|\vec{r}_i - \vec{r}_j|} + \frac{\vec{P}_i\cdot(\vec{r}_i - \vec{r}_j)\,\vec{P}_j\cdot(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^3}\right\}$$

For simplicity, we consider the Coulomb interaction only, thus:

$$\mathcal{H}_{int} \equiv V = \frac{1}{2}\sum_i\sum_j{}' \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$$

Then

$$\frac{\partial V}{\partial \vec{P}_i} = 0, \quad \frac{\partial V}{\partial \vec{r}_i} = \sum_j{}' \frac{\partial}{\partial \vec{r}_i}\frac{e^2}{|\vec{r}_i - \vec{r}_j|} = -e^2\sum_j{}' \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

$$\frac{\partial \mathcal{H}_o}{\partial \vec{P}_i} = \frac{1}{m}(\vec{P}_i - \tfrac{e}{c}\vec{A}_i)$$

$$\frac{\partial \mathcal{H}_o}{\partial \vec{r}_i} = \frac{1}{m}\left[\vec{P}_i\times(\frac{\partial}{\partial \vec{r}_i}\times\vec{P}_i) + (\vec{P}_i\cdot\frac{\partial}{\partial \vec{r}_i})\vec{P}_i\right] + e\frac{\partial \varphi_i}{\partial \vec{r}_i}$$

where

$$\vec{\nabla}_i \equiv \frac{\partial}{\partial \vec{r}_i} \;, \quad \vec{P}_i = \vec{p}_i - \frac{e}{c}\vec{A}_i$$

$$\vec{A}_i \equiv \vec{A}(\vec{r}_i\, t)\,, \quad \varphi_i \equiv \varphi(\vec{r}_i\, t)$$

Substituting these into the S-particle Liouville's equation we have

$$\frac{\partial F^{(S)}}{\partial t} + \sum_{i=1}^{S}\Big\{\frac{1}{m}(\vec{p}_i - \frac{e}{c}\vec{A}_i)\frac{\partial F^{(S)}}{\partial \vec{r}_i} - \frac{\partial F^{(S)}}{\partial \vec{p}_i}\Big[e\vec{\nabla}_i\varphi_i + \frac{1}{m}(\vec{p}_i - \frac{e}{c}\vec{A}_i)\times\vec{\nabla}_i\times(\vec{p}_i - \frac{e}{c}\vec{A}_i)$$

$$+ \frac{1}{m}(\vec{p}_i - \frac{e}{c}\vec{A}_i)\cdot\frac{\partial}{\partial \vec{r}_i}(\vec{p}_i - \frac{e}{c}\vec{A}_i)\Big]\Big\}$$

$$= \sum_{i=1}^{S}\int \prod_{k=S+1}^{N} d^6\zeta_k \Big[\frac{\partial}{\partial \vec{r}_i}\sum_{ij}^{N}{}' \frac{1}{2}\frac{e^2}{|\vec{r}_i - \vec{r}_j|}\Big]\frac{\partial}{\partial \vec{p}_i} F^{(N)}(t)$$

$$= \sum_{i=1}^{S}\int \prod_{k=S+1}^{N} d^6\zeta_k \Big[\frac{\partial}{\partial \vec{r}_i}\sum_{j=1}^{N}{}' \frac{e^2}{|\vec{r}_i - \vec{r}_j|}\Big]\frac{\partial}{\partial \vec{p}_i} F^{(N)}(t)$$

$$= \sum_{i=1}^{S}\int \prod_{k=S+1}^{N} d^6\zeta_k \Big[\sum_{j=1}^{S}{}' \frac{\partial}{\partial \vec{r}_i}\Big(\frac{e^2}{r_{ij}}\Big) + \sum_{j=S+1}^{N}\frac{\partial}{\partial \vec{r}_i}\Big(\frac{e^2}{r_{ij}}\Big)\Big]\frac{\partial}{\partial \vec{p}_i} F^{(N)}(t)$$

$$= \sum_{i=1}^{S}\int \prod_{k=S+1}^{N} d^6\zeta_k \frac{\partial F^{(N)}}{\partial \vec{p}_i}\sum_{j=1}^{S}{}' \frac{\partial}{\partial \vec{r}_i}\Big(\frac{e^2}{r_{ij}}\Big) + \sum_{i=1}^{S}\int \prod_{k=S+1}^{N} d^6\zeta_k \frac{\partial F^{(N)}}{\partial \vec{p}_i}\sum_{j=S+1}^{N}{}' \frac{\partial}{\partial \vec{r}_i}\Big(\frac{e^2}{r_{ij}}\Big)$$

$$= \sum_{i=1}^{S}\sum_{j=1}^{S}{}' \frac{\partial}{\partial \vec{r}_i}\Big(\frac{e^2}{r_{ij}}\Big)\frac{\partial}{\partial \vec{p}_i}\int \prod_{k=S+1}^{N} d^6\zeta_k\, F^{(N)}(t) + \sum_{i=1}^{S}\int \prod_{k=S+1}^{N} d^6\zeta_k (N-S)\frac{\partial}{\partial \vec{r}_i}\frac{e^2}{|\vec{r}_i - \vec{r}_{S+1}|}\frac{\partial}{\partial \vec{p}_i}F^{(N)}$$

$$= \sum_{i=1}^{S}\sum_{j=1}^{S}{}' \frac{-e^2\vec{r}_{ij}}{r_{ij}^3}\cdot\frac{\partial}{\partial \vec{p}_i}F^{(S)} + (N-S)\sum_{i=1}^{S}\int d^6\zeta_{S+1}\frac{\partial}{\partial \vec{r}_i}\Big(\frac{e^2}{|\vec{r}_i - \vec{r}_{S+1}|}\Big)\int \prod_{k=S+2}^{N} d^6\zeta_k \frac{\partial}{\partial \vec{p}_i}F^{(N)}$$

$$\cong e^2\sum_{i=1}^{S}\sum_{j=1}^{S}{}' \frac{\partial}{\partial \vec{r}_i}\Big(\frac{1}{|\vec{r}_i - \vec{r}_j|}\Big)\cdot\frac{\partial}{\partial \vec{p}_i}F^{(S)} + Ne^2\sum_{i=1}^{S}\int d^6\zeta_{S+1}\frac{\partial}{\partial \vec{r}_i}\Big(\frac{1}{|\vec{r}_i - \vec{r}_{S+1}|}\Big)\frac{\partial}{\partial \vec{p}_i}F^{(S+2)}(t)$$

Now

$$\vec{p}_i - \frac{e}{c}\vec{A}_i = m\vec{v}_i \,, \quad \vec{E}_i = -\frac{\partial}{\partial \vec{r}_i}\varphi_i - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}_i \,, \quad \vec{B}_i = \frac{\partial}{\partial \vec{r}_i}\times \vec{A}_i$$

$$e\vec{E}_i = -e\vec{\nabla}_i\varphi_i - \frac{e}{c}\frac{\partial}{\partial t}\vec{A}_i \,, \quad \vec{A}_i \equiv \vec{A}(\vec{r}_i, t) \,, \quad \varphi_i \equiv \varphi(\vec{r}_i, t) \,,$$

$$e\vec{E}(\vec{r}_i, t) + \frac{e}{c}\frac{\partial}{\partial t}\vec{A}(\vec{r}_i, t) = -e\vec{\nabla}_i\phi(\vec{r}_i, t)$$

Thus we have

$$\frac{\partial F^{(S)}}{\partial t} + \sum_{i=1}^{S}\left\{\vec{v}_i \cdot \frac{\partial}{\partial \vec{r}_i}F^{(S)} + \left[e\vec{E}_i + \frac{e}{c}\vec{v}_i\times\vec{B}_i + \frac{e}{c}\left[\frac{\partial \vec{A}_i}{\partial t} + \vec{v}_i\cdot\vec{\nabla}_i\vec{A}_i\right]\right]\cdot \frac{\partial F^{(S)}}{\partial \vec{p}_i}\right\} = e^2\sum_{ij}^{S}{}'\left(\frac{\partial}{\partial \vec{r}_i}\frac{1}{|\vec{r}_i-\vec{r}_j|}\right)\cdot\frac{\partial F^{(S)}}{\partial \vec{p}_i} + Ne^2\sum_{i=1}^{S}\int d^6\zeta_{S+1}\left[\frac{\partial}{\partial \vec{r}_i}\frac{1}{|\vec{r}_i-\vec{r}_{S+1}|}\right]\cdot\frac{\partial}{\partial \vec{p}_i}F^{(S+2)}$$

where

$$F^{(S+2)} = \int \prod_{\lambda=S+2}^{N} d^6\zeta_\lambda \; F^{(N)}(t)$$

We now change variables from $\vec{r}_i, \vec{p}_i, t$ to $\vec{r}_i, \vec{v}_i, t$ and note that

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} - \frac{e}{m}\frac{\partial \vec{A}_i}{\partial t}\cdot\frac{\partial}{\partial \vec{v}_i} \,, \quad \vec{v}_i\cdot\frac{\partial}{\partial \vec{r}_i} \longrightarrow \vec{v}_i\cdot\frac{\partial}{\partial \vec{r}_i} - \frac{e}{m}\left[\left(\vec{v}_i\cdot\frac{\partial}{\partial \vec{r}_i}\right)\vec{A}_i\right]\cdot\frac{\partial}{\partial \vec{v}_i} \,, \quad \frac{\partial}{\partial \vec{p}_i} \longrightarrow \frac{1}{m}\frac{\partial}{\partial \vec{v}_i}$$

Performing the indicated change of variables we obtain the non-relativistic Born-Green Hierarchy:

$$\frac{\partial F^{(S)}}{\partial t} + \sum_{i=1}^{S}\left\{\vec{v}_i\cdot\frac{\partial}{\partial \vec{r}_i} + \frac{e}{m}\left[\vec{E}(\vec{r}_i\, t) + \frac{1}{c}\vec{v}_i\times\vec{B}(\vec{r}_i\, t)\right]\cdot\frac{\partial}{\partial \vec{v}_i}\right\}F^{(S)}$$

$$= \frac{e^2}{m}\sum_{ij}^{S}{}'\left[\frac{\partial}{\partial \vec{r}_i}\left(\frac{1}{|\vec{r}_i-\vec{r}_j|}\right)\right]\cdot\frac{\partial F^{(S)}}{\partial \vec{v}_i} + \frac{Ne^2}{m}\sum\int d^3r_{S+1}\, d^3v_{S+1}\left[\frac{\partial}{\partial \vec{r}_i}\left(\frac{1}{|r_i - r_{S+1}|}\right)\right]\cdot\frac{\partial F^{(S+1)}}{\partial \vec{v}_i} \qquad 5\text{-}1$$

V-2 The B-B-G-K-Y Hierarchy in Relativistic Form

We have shown in the last chapter that the S-particle distribution function satisfies the equation

$$\frac{\partial F^{(S)}}{\partial t} + \sum_{i=1}^{S}\left\{ \frac{\partial F^{(S)}}{\partial \vec{r}_i}\frac{\partial \mathcal{H}_o}{\partial \vec{p}_i} - \frac{\partial F^{(S)}}{\partial \vec{p}_i}\frac{\partial \mathcal{H}_o}{\partial \vec{r}_i}\right\} = \sum_{i=1}^{S}\left\{ \frac{\partial V}{\partial \vec{r}_i}\frac{\partial F^{(S)}}{\partial \vec{p}_i} - \frac{\partial V}{\partial \vec{p}_i}\frac{\partial F^{(S)}}{\partial \vec{r}_i}\right\}.$$

Consider the following Hamiltonian: $\mathcal{H} = \mathcal{H}_o + \mathcal{H}_{int}$, where

$$\mathcal{H}_o = \sum_{i=1}^{N}\left\{ c\sqrt{m_o^2c^2 + (\vec{p}_i - \tfrac{e}{c}\vec{A}_i)^2} + e\varphi(\vec{r}_i\, t)\right\}$$

$$\mathcal{H}_{int} = \sum_{i=1}^{N}\sum_{j=1}^{N}{}' \frac{1}{2}\frac{e^2}{|\vec{r}_i-\vec{r}_j|} - \frac{1}{2}\frac{e^2}{2m_o^2c^2}\sum_{i=1}^{N}\sum_{j=1}^{N}{}'\left\{ \frac{\vec{p}_i\cdot\vec{p}_j}{|\vec{r}_i-\vec{r}_j|} + \frac{\vec{p}_i\cdot\vec{r}_i-\vec{r}_j\;\vec{p}_j\,\vec{r}_i-\vec{r}_j}{|\vec{r}_i-\vec{r}_j|^3}\right\},$$

i.e. $\mathcal{H}_{int} = V_{ij} + U_{ij}(\vec{r}_i, \vec{r}_j, \vec{p}_i, \vec{p}_j)$ where

$$V_{ij} = \frac{1}{2}\sum_{ij}^{N}{}' \frac{e^2}{|\vec{r}_i-\vec{r}_j|}, \quad U_{ij} = -\frac{e^2}{4m_o^2c^2}\sum_{ij}^{N}{}'\left\{ \frac{\vec{p}_i\cdot\vec{p}_j}{r_{ij}} + \frac{\vec{p}_i\cdot\vec{r}_{ij}\,\vec{p}_j\cdot\vec{r}_{ij}}{r_{ij}^3}\right\}.$$

For simplicity, we again neglect $U_{ij}(\vec{r}_i, \vec{r}_j, \vec{p}_i, \vec{p}_j)$. Now

$$\frac{\partial \mathcal{H}_o}{\partial \vec{r}_i} = e\vec{\nabla}_i\varphi_i + \frac{1}{\gamma_i m_o}\left[\vec{P}_i \times \vec{\nabla}_i \times \vec{P}_i + \vec{P}_i\cdot\vec{\nabla}_i\,\vec{P}_i\right], \quad \vec{P}_i = \vec{p}_i - \frac{e}{c}\vec{A}_i,$$

$$\gamma_i = \frac{1}{\sqrt{1-\frac{v_i^2}{c^2}}}, \quad \frac{\partial V_{ij}}{\partial \vec{p}_i} = 0, \quad \frac{\partial V_{ij}}{\partial \vec{r}_i} = \frac{\partial}{\partial \vec{r}_i}\sum_j{}' \frac{e^2}{|\vec{r}_i-\vec{r}_j|}$$

$$\frac{\partial \mathcal{H}_o}{\partial \vec{p}_i} = \frac{\vec{P}_i}{\gamma_i m_o}, \quad \sqrt{m_o^2c^2 + P_i^2} = \gamma_i\, m_o c$$

Therefore

$$\frac{\partial F^{(S)}}{\partial t}+\sum_{i=1}^{S}\left\{\frac{\vec{P}_i}{\gamma_i m_0}\cdot\frac{\partial F^{(S)}}{\partial \vec{r}_i}-\frac{\partial F^{(S)}}{\partial \vec{p}_i}\cdot\left[e\frac{\partial \varphi_i}{\partial \vec{r}_i}+\frac{1}{\gamma_i m_0}\left(\vec{P}_i\times\vec{\nabla}_i\times\vec{P}_i+\vec{P}_i\cdot\vec{\nabla}_i\vec{P}_i\right)\right]\right\}$$

$$=\sum_{i=1}^{S}\left\{\left[\frac{\partial}{\partial \vec{r}_i}\sum{}' \frac{e^2}{|\vec{r}_i-\vec{r}_j|}\right]\cdot\frac{\partial F^{(S)}}{\partial \vec{p}_i}\right\},$$

$$\frac{\partial F^{(S)}}{\partial t}+\sum_{i=1}^{S}\left\{\frac{1}{\gamma_i m_0}\vec{P}_i\cdot\frac{\partial F^{(S)}}{\partial \vec{r}_i}-\frac{\partial F^{(S)}}{\partial \vec{p}_i}\cdot\left[e\frac{\partial \varphi_i}{\partial \vec{r}_i}+\frac{1}{\gamma_i m_0}\left(\vec{P}_i\times\frac{\partial}{\partial \vec{r}_i}\times\vec{P}_i+\vec{P}_i\cdot\frac{\partial}{\partial \vec{r}_i}\vec{P}_i\right)\right]\right\}$$

$$=\sum_{i=1}^{S}\left\{\frac{\partial}{\partial \vec{r}_i}\left[\left(\sum_{j=1}^{S}{}'+\sum_{j=S+1}^{N}{}'\right)\frac{e^2}{|\vec{r}_i-\vec{r}_j|}\right]\cdot\frac{\partial}{\partial \vec{p}_i}\right\}\int\prod_{\lambda=S+1}^{N} d^6\zeta_\lambda\; F^{(N)}(t),$$

$$\frac{\partial F^{(S)}}{\partial t}+\sum_{i=1}^{S}\left\{\frac{1}{\gamma_i m_0}\vec{P}_i\cdot\frac{\partial F^{(S)}}{\partial \vec{r}_i}+\left[e\vec{E}(\vec{r}_i t)+\frac{ce\vec{P}_i\times\vec{B}(\vec{r}_i t)}{\gamma_i m_0 c}\right]\cdot\frac{\partial F^{(S)}}{\partial \vec{p}_i}\right\}$$

$$=\sum_{ij}^{S}{}'\left[\frac{\partial}{\partial \vec{r}_i}\left(\frac{e^2}{|\vec{r}_i-\vec{r}_j|}\right)\right]\cdot\frac{\partial F^{(S)}}{\partial \vec{p}_i}+(N-S)\sum_{i=1}^{S}\int d^3r_{S+1}\,d^3p_{S+1}\left[\frac{\partial}{\partial \vec{r}_i}\left(\frac{e^2}{|\vec{r}_i-\vec{r}_{S+1}|}\right)\right]\cdot\frac{\partial}{\partial \vec{p}_i}\int\prod_{\lambda=S+2}^{N} d^6\zeta_\lambda\, F^{(N)}(t)$$

We can also express the above equation in terms of $\vec{r}_i, \vec{v}_i, t$ as follows:

$$\frac{\partial F^{(S)}}{\partial t}+\sum_{i=1}^{S}\left\{\vec{v}_i\cdot\frac{\partial F^{(S)}}{\partial \vec{r}_i}+\frac{e}{m_0}\sqrt{1-\frac{v_i^2}{c^2}}\left[\vec{E}(\vec{r}_i,t)+\frac{1}{c}\vec{v}_i\times\vec{B}(\vec{r}_i,t)-\frac{1}{c^2}\vec{v}_i\cdot\vec{E}_i\vec{v}_i\right]\cdot\frac{\partial F^{(S)}}{\partial \vec{v}_i}\right\}$$

$$=\sum_{ij}^{S}{}'\left[\frac{\partial}{\partial \vec{r}_i}\left(\frac{e^2}{|\vec{r}_i-\vec{r}_j|}\right)\right]\cdot\frac{\partial F^{(S)}}{\partial \vec{v}_i}+\frac{Ne^2}{m_0}\sum_{i=1}^{S}\int d^3r_{S+1}\,d^3v_{S+1}\left[\frac{\partial}{\partial \vec{r}_i}\left(\frac{1}{|r_i-r_{S+1}|}\right)\right]\cdot\frac{\partial}{\partial \vec{v}_i}F^{(S+1)}.$$

We shall derive the Rosenbluth-Rostoker Hierarchy on the basis of this equation in the next section.

V-3 The Rostoker-Rosenbluth Expansion, and Corrections to the Boltzmann-Vlasov Equation

We shall examine more critically the assumptions made in the derivation of the Boltzmann-Vlasov Equation. The assumption that Liouville's equation is applicable to the physical system under consideration is hardly open to question. However, the independent trajectory assumption is not on a firm basis, and we shall investigate in detail the criteria of validity of this assumption (i.e., $F^{(N)} = \prod_{j=1}^{N} f_j$) as well as explicit corrections to the Boltzmann-Vlasov equation.

It is clear by now that in contrast to the neutral dilute gas case, we can anticipate the Ansatz of independent trajectory (i.e., $F^{(N)} = \prod_{j=1}^{N} f_j$) to be valid provided only that the parameter α,

$$\alpha = \left[\frac{k}{4\pi e^2}\right]^{3/2} T^{3/2}\, n^{-1/2}$$

is very large compared to unity. Following Rosenbluth and Rostoker, we shall expand the solution of Liouville's equation in a power series in α^{-1}, and thereby obtain corrections to the Boltzmann-Vlasov equation. In order that the significant features will not be obscured in a morass of algebra, we shall consider a plasma that consists of only one species, with Coulomb interactions and without external fields present. For this purpose, we shall redefine the S-particle distribution function by the formula

$$F^{(S)}(t) = V^{S} \int \prod_{j=S+1}^{N} d^6 \zeta_j \, F^{(N)}(t)$$

In terms of this new definition of $F^{(S)}(t)$, equation 5-1 of Section 1 can be cast into the form

$$\frac{\partial F^{(S)}}{\partial t} + \sum_{i=1}^{S} \vec{v}_i \cdot \frac{\partial F^{(S)}}{\partial \vec{r}_i} + \frac{e^2}{m} \sum_{i=1}^{S} \sum_{j=1}^{S}{}' \vec{E}_{ij} \cdot \frac{\partial F^{(S)}}{\partial \vec{v}_i}$$

$$+ \frac{ne^2}{m} \sum_{j=1}^{S} \int d^3 r_{S+1}\, d^3 p_{S+1}\, \vec{E}_{j,S+1} \cdot \frac{\partial}{\partial \vec{v}_j} F^{(S+1)} = 0$$

One procedure, which is equivalent to making a power series expansion in terms of α^{-1}, involves taking the limits (i.e. the fluid limits)

$$\left\{ \begin{array}{c} e \longrightarrow 0 \\ m \longrightarrow 0 \\ \frac{N}{V} = n \longrightarrow \infty \end{array} \right\} \text{ and } \left\{ \begin{array}{c} ne \\ nm \\ \frac{e}{m} \end{array} \right\} \text{ remain finite}$$

and write

$$F^{(S)} = F_0^{(S)} + F_1^{(S)} + \cdots$$

where $F_1^{(S)}$ is of order α^{-1} relative to $F_0^{(S)}$. On substituting this assumed form for $F^{(S)}$ into equation ( ), we obtain

$$\frac{\partial F_0^{(S)}}{\partial t} + \sum_{i=1}^{S} \vec{v}_i \cdot \frac{\partial F_0^{(S)}}{\partial \vec{r}_i} + \frac{ne^2}{m} \sum_{i=1}^{S} \int d^3 r_{S+1}\, d^3 p_{S+1}\, \vec{E}_{i,S+1} \cdot \frac{\partial F_0^{(S+1)}}{\partial \vec{v}_i} = 0$$

$$\frac{\partial F_1^{(S)}}{\partial t} + \sum_{i=1}^{S} \vec{v}_i \cdot \frac{\partial F_1^{(S)}}{\partial \vec{r}_i} + \frac{ne^2}{m} \sum_{i=1}^{S} \int d^3 r_{S+1}\, d^3 p_{S+1}\, \vec{E}_{i,S+1} \cdot \frac{\partial F_1^{(S+1)}}{\partial \vec{v}_i}$$

$$+ \frac{e^2}{m} \sum_{i=1}^{S} \sum_{j=1}^{S}{}' \vec{E}_{ij} \frac{\partial F_0^{(S)}}{\partial \vec{v}_i} = 0$$

We note that in the lowest order of approximation $F_0^{(S)}$ (i.e., the S-particle distribution function in the zeroth order of approximation), $\alpha \to \infty$, so all of the particles in the plasma are within a Debye radius of each other and consequently no particle is shielded from the Coulomb field of any other one. Thus, each particle experiences forces due to all of the other particles in the plasma, and the few-particle collisions are completely negligible compared to the Coulomb forces produced by the overwhelmingly large number of particles without shielding. In this approximation, we may assert

$$F_0^{(S)} = \prod_{i=1}^{S} f_0(\vec{r}_i, \vec{v}_i, t)$$

where f_0 is the single particle distribution function in the limit $\alpha \to \infty$. Although this assumption is very similar to the Ansatz of independent trajectory, it is on a completely different basis. Since in the present formulation, the approximation of $F_0^{(S)} = \prod_{i=1}^{S} f_0(\vec{r}_i, \vec{p}_i, t)$ is only the first term in a power series of α^{-1}, for which higher-order corrections can be obtained in a systematic way; while on the other hand the assumption of independent trajectory used in connection with the derivation of the Vlasov-Boltzmann equation (from Liouville's theorem) implies no further corrections.

The equation satisfied by f_0 is clearly given by

$$\sum \left\{ \left[\prod_{j=1}^{S}{}' f_0(\vec{r}_j \, \vec{v}_j \, t) \right] \left[\frac{\partial}{\partial t} + \vec{v}_i \cdot \frac{\partial}{\partial \vec{r}_i} + \frac{ne^2}{m} \int d^3 r_{S+1} \, d^3 p_{S+1} \, f_0(\vec{r}_{S+1}, \vec{p}_{S+1}, t) \vec{E}_{i,S+1} \cdot \frac{\partial}{\partial \vec{v}_i} \right] f_0(\vec{r}_i, \vec{v}_i, t) \right\} = 0$$

So that

$$\frac{\partial f_0}{\partial t} + \vec{v}\cdot\frac{\partial f_0}{\partial \vec{r}} + \frac{e}{m}\vec{E}_0\cdot\frac{\partial f_0}{\partial \vec{v}} = 0$$

where

$$\vec{E}_0(\vec{r},t) = -ne\int d^3r'\, d^3v'\, f_0(\vec{r}',\vec{v}',t)\frac{\partial}{\partial \vec{r}}\frac{1}{|\vec{r}-\vec{r}'|} \quad .$$

The first order correction to the Vlasov-Boltzmann equation can be obtained by considering the solution of the equation

$$\frac{\partial F_1^{(s)}}{\partial t} + \sum_{i=1}^{s}\vec{v}_i\cdot\frac{\partial}{\partial \vec{r}_i}F_1^{(s)} + \frac{ne^2}{m}\sum_{i=1}^{s}\int d^3r_{s+1}\, d^3p_{s+1}\,\vec{E}_{i,s+1}\cdot\frac{\partial}{\partial \vec{v}_i}F_1^{(s+1)}$$
$$+ \frac{e^2}{m}\sum_{i=1}^{s}{\sum_{j=1}^{s}}' \vec{E}_{ij}\cdot\frac{\partial}{\partial \vec{v}_i}F_0^{(s)} = 0 \qquad 5\text{-}3\text{-}1$$

Consider first the simple case of a two-particle distribution function $F^{(2)}(\vec{r}_1,\vec{v}_1;\vec{r}_2,\vec{v}_2;t)$:

$$F_0^{(2)} = \prod_{i=1}^{2} f_0(x_i,t) = f_0(x_1,t)f_0(x_2,t), \quad x_i \equiv (\vec{r}_i,\vec{v}_i)$$

$$F^{(2)} = \left[f_0(x_1,t)+\epsilon f_1(x_1,t)\right]\left[f_0(x_2,t)+\epsilon f_1(x_2,t)+\epsilon P(x_1,x_2,t)\right]$$
$$= f_0(x_1,t)f_0(x_2,t) + \epsilon\left[f_1(x_1,t)f_0(x_2,t) + f_0(x_1,t)f_1(x_2,t)\right.$$
$$\left. + f_0(x_1,t)P(x_1,x_2,t)\right] + O(\epsilon^2)$$
$$= F_0^{(2)} + \epsilon F_1^{(2)} + \epsilon^2 F_2^{(2)}, \qquad \epsilon = O\left(\frac{1}{\alpha}\right)$$
$$\therefore\ F_1^{(2)} = f_0(x_1,t)f_1(x_2,t) + f_1(x_1,t)f_0(x_2,t) + f_0(x_1,t)P(x_1,x_2,t)$$

This example suggests that we may try the Ansatz (for 1st order correction of S-particle distribution)

$$F_1^{(S)} = \sum_{i=1}^{S} \left\{ {\prod_{j=1}^{S}}' f_0(x_j\, t) \right\} f_1(x_i\, t) + \frac{1}{2} {\sum_{i=1}^{S} \sum_{j=1}^{S}}' P(x_i, x_j, t) \left\{ {\prod_{k=1}^{S}}'' f_0(x_k, t) \right\}$$

where the primes and double primes mean omit the terms $i = j$, and $i = k$, $j = k$ respectively.

The equation satisfied by f_1 and P can be obtained by straightforward substitution of the above Ansatz into the equation satisfied by $F_1^{(S)}$ for the two special cases of S = 1, and S = 2.

For the case S = 1, we obtain

$$\frac{\partial}{\partial t} f_1(x_1\, t) + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{r}_1} f_1 + \frac{e}{m}\left[\vec{E}_0 \cdot \frac{\partial}{\partial \vec{v}_1} f_1 + \vec{E}_1 \cdot \frac{\partial}{\partial \vec{v}_1} f_0 \right]$$
$$+ \frac{ne^2}{m} \int d^3 r_2\, d^3 v_2\, \vec{E}_{12} \cdot \frac{\partial}{\partial \vec{v}_1} P(x_1, x_2, t) = 0$$

where

$$\vec{E}_0(\vec{r}, t) = -ne \int d^3 r'\, d^3 v'\, f_0(\vec{r}', \vec{v}', t) \frac{\partial}{\partial \vec{r}} \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\vec{E}_1(\vec{r}, t) = -ne \int d^3 r'\, d^3 v'\, f_1(\vec{r}', \vec{v}', t) \frac{\partial}{\partial \vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} .$$

For the case S = 2, we obtain

$$\left\{\frac{\partial}{\partial t}+\vec{v}_1\cdot\frac{\partial}{\partial\vec{r}_1}+\vec{v}_2\cdot\frac{\partial}{\partial\vec{r}_2}+\frac{e}{m}\left[\vec{E}_0(\vec{r}_1,t)\frac{\partial}{\partial\vec{v}_1}+\vec{E}_0(\vec{r}_2,t)\frac{\partial}{\partial\vec{v}_2}\right]\right\}P(x_1,x_2,t)$$

$$+\frac{ne^2}{m}\left\{\int d^3r_3\,d^3v_3\,\vec{E}_{13}\,P(x_2,x_3,t)\cdot\frac{\partial}{\partial\vec{v}_1}f_0(x_1,t)+(1\leftarrow\rightarrow 2)\right\}$$

$$+\frac{e^2}{m}\left\{f_0(x_1,t)\vec{E}_{21}\cdot\frac{\partial}{\partial\vec{v}_2}f_0(x_2,t)+(1\longleftrightarrow 2)\right\}=0$$

where $(1\leftarrow\rightarrow 2)$ means repeat the preceding term but with the subscripts 1 and 2 interchanged and where $\vec{E}_{ij}$ is given by

$$\vec{E}_{ij}=-\frac{\partial}{\partial\vec{r}_i}\left(\frac{1}{|\vec{r}_i-\vec{r}_j|}\right)=\frac{\vec{r}_i-\vec{r}_j}{|\vec{r}_i-\vec{r}_j|^3}\ .$$

V-4 <u>Summary</u>

The development of plasma dynamics has followed two different approaches. The approach we presented here so far follows that of Born-Bogoliubov-Green-Kirkwood-Yvon, and Rostoker-Rosenbluth. In this formalism, Liouville's equation is first integrated in terms of the S-particle distribution function, $(S \ll N)$ and the result is the familiar B-B-G-K-Y Hierarchy which relates the S-particle distribution function to the (S+1)-particle distribution function. Rostoker and Rosenbluth have developed a series expansion in terms of the parameter α^{-1} (where $\alpha = n\delta^3$, $\delta=\sqrt{\frac{kT}{4\pi ne^2}}$) by taking the fluid limit. We have seen that the lowest order of approximation in the fluid limit gives the Vlasov equation, and the α^{-1} correction is given by Eq. 5-3-1 in terms of two particle correlation function. Another approach is that due to Prigogine, Balescu, etc., of the Belgian school. These authors also start their analysis from Liouville's equation; however, in contrast to the approach we have followed, they first integrate the many particle Liouville's equation by the method of Fourier analysis and Green's functions. The solution of this N-particle Liouville equation is then expanded by a diagramatic method similar to that used in field theory and virial expansions. The few-particle distribution function is obtained by integrating the solution of the N-particle Liouville equation. The generalization to include quantum effects is done by using the Von Neumann density matrix and Wigner representation. The first order perturbation calculation by using the diagramatic method gives the Landau form of the Fokker-Plank equation, which is obtained by the replacement of the zero on the right-hand side of the Vlasov equation with a collision term. These Fokker-Plank terms are not associated with the smoothed-out microscopic fields already included on the left-hand side of the Boltzmann-Vlasov equation, nor are they associated with the violent, close-encounter binary collisions usually described by the Boltzmann collision integrals. But rather, these Fokker-Plank terms are based on a mechanism that lies between these two (i.e. the discrete nature of the close-encounter collisions and the continuum nature in the fluid limit). The mechanism for these distant-encounter collisions (Fokker-Plank terms) may be interpreted as producing changes in the distribution function as a result of gradual alterations in the particle trajectories due to many small-momentum-transfer or large-impact-parameter (and hence small angle) scatterings. Since such scatterings can occur only for particles within a Debye sphere, it is clear that the Fokker-Plank equation is equivalent to our α^{-1} corrections. This equivalence has been demonstrated by Bernstein and Trehan.

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