Solutions to Practice Problems

Exercise 4.12
Prove that the sequence $\{\cos(n\pi)\}_{n=1}^{\infty}$ is divergent.

Solution.
Note that $\{\cos(n\pi)\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ and by Exercise 4.4, this sequence is divergent $\blacksquare$

Exercise 4.13
Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined by $a_n = n$ for all $n \in \mathbb{N}$. Explain why the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge to any limit.

Solution.
The sequence is unbounded $\blacksquare$

Exercise 4.14
(a) Show that for all $n \in \mathbb{N}$ we have

$$\frac{n!}{n^n} \le \frac{1}{n}.$$

(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{n!}{n^n}$ is convergent and find its limit.

Solution.
(a) We know that $\frac{n-i}{n} \le 1$ for all $0 \le i \le n-1$. Thus, $\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{n\cdot n\cdots n} = \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{2}{n}\cdot\frac{1}{n} \le \frac{1}{n}$.
(b) By the Squeeze rule we find that $\lim_{n\to\infty}\frac{n!}{n^n} = 0$ $\blacksquare$

Exercise 4.15
Using only the definition of convergence show that

$$\lim_{n\to\infty}\frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} = 1.$$

Solution.
Let $\epsilon > 0$. We want to find a positive integer N such that if $n \ge N$ then

$$\left|\frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} - 1\right| < \epsilon$$

or

$$\left|\frac{-4001}{\sqrt[3]{n}-1001}\right| < \epsilon.$$

Let $n > 1001^3$. Then $\sqrt[3]{n}-1001 > 0$ so that the previous inequality becomes

$$\frac{4001}{\sqrt[3]{n}-1001} < \epsilon.$$

Solving this for n we find

$$n > \left(\frac{4001}{\epsilon} + 1001\right)^3.$$

Let N be a positive integer greater than $\left(\frac{4001}{\epsilon}+1001\right)^3$. Then for $n \geq N$ we have

$$\left|\frac{\sqrt[3]{n}-5001}{\sqrt[3]{n}-1001} - 1\right| < \epsilon \blacksquare$$

Exercise 4.16
Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2+a_n}$ for all $n \in \mathbb{N}$. Show that $\{a_n\}_{n=1}^{\infty}$ is bounded. Hint: Exercise 1.14.

Solution.
The proof is by induction on n. For $n = 1$ we have $a_1 = 1 \leq 2$. Suppose that $a_n \leq 2$. Then $a_{n+1} = \sqrt{2+a_n} \leq \sqrt{2+2} = 2 \blacksquare$

Exercise 4.17
Calculate $\lim_{n\to\infty} \frac{(n^2+1)\cos n}{n^3}$ by using the squeeze rule.

Solution.
We have

$$-\frac{n^2+1}{n^3} \leq \frac{(n^2+1)\cos n}{n^3} \leq \frac{n^2+1}{n^3}.$$

By the Squeeze rule we conclude that the limit is 0 $\blacksquare$

Exercise 4.18
Calculate $\lim_{n\to\infty} \frac{2(-1)^{n+3}}{\sqrt{n}}$ by using the squeeze rule.

Solution.
We have

$$-\frac{2}{\sqrt{n}} \leq \frac{2(-1)^{n+3}}{\sqrt{n}} \leq \frac{2}{\sqrt{n}}.$$

By the Squeeze rule the limit is 0 ■

Exercise 4.19
Suppose that $\lim_{n\to\infty} a_n = L$ with $L > 0$. Show that there is a positive integer N such that $2a_N > L$.

Solution.
Let $\epsilon = \frac{L}{2}$. Then there is a positive integer N such that if $n \geq N$ we have $|a_n - L| < \frac{L}{2}$. Thus, $|a_N - L| < \frac{L}{2}$ or $-\frac{L}{2} < a_N - L < \frac{L}{2}$. Hence, $a_N > \frac{L}{2}$ or $2a_N > L$ ■

Exercise 4.20
Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Clearly, $a < a + \frac{1}{n}$.
(a) Show that there is $a_n \in \mathbb{Q}$ such that $a < a_n < a + \frac{1}{n}$.
(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a.
We have proved that if a is a real number then there is a sequence of rational numbers converging to a. We say that the set $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Solution.
(a) This follows from Exercise 3.6(c).
(b) Applying the Squeeze rule, we obtain $\lim_{n\to\infty} a_n = a$ ■

Exercise 4.21
Consider the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \alpha$ for all $n \in \mathbb{N}$. Use the definition of convergence to show that $\lim_{n\to\infty} a_n = \alpha$.

Solution.
Let $\epsilon > 0$ be given. Let N be any positive integer. Then for $n \geq N$ we have $|a_n - \alpha| = |\alpha - \alpha| = 0 < \epsilon$ ■

Exercise 4.22
Suppose that $\lim_{n\to\infty} a_n = L$. For each $n \in \mathbb{N}$ let $b_n = \frac{a_n + a_{n+1}}{2}$. Show that $\lim_{n\to\infty} b_n = L$.

Solution.
Let $\epsilon > 0$ be given. There is a positive integer N such that $|a_n - L| < \epsilon$ for all $n \geq N$. Since $n+1 \geq N+1 \geq N$ we have $|a_{n+1} - L| < \epsilon$. Hence, for all $n \geq N$ we have

$$|b_n - L| = \left|\frac{a_n - L}{2} + \frac{a_{n+1} - L}{2}\right| \leq \left|\frac{a_n - L}{2}\right| + \left|\frac{a_{n+1} - L}{2}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{n\to\infty} b_n = L$ ■

Exercise 4.23
(a) Show that if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} |a_n| = |L|$.
(b) Give an example of a sequence where $\{a_n\}_{n=1}^{\infty}$ is divergent but $\{|a_n|\}_{n=1}^{\infty}$ is convergent.

Solution.
(a) Let $\epsilon > 0$ be given. Then there is a positive integer N such that $|a_n - L| < \epsilon$ for all $n \geq N$. Thus,

$$||a_n| - |L|| \leq |a_n - L| < \epsilon$$

for all $n \geq N$. This shows that $\lim_{n\to\infty} |a_n| = |L|$.
(b) The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent but $\{|a_n|\}_{n=1}^{\infty} = \{1, 1, 1, \cdots\}$ is convergent with limit 1 ■

Exercise 4.24
Show that if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Solution.
This follows from $-|a_n| \leq a_n \leq |a_n|$ and the squeeze rule ■

Exercise 4.25
Let $a = \sup A$. Show that there is a sequence $\{a_n\}_{n=1}^{\infty} \subset A$ such that $\lim_{n\to\infty} a_n = a$. Hint: Exercise 3.12.

Solution.
By Exercise 3.12, for each $n \in \mathbb{N}$ we can find $a_n \in A$ such that $0 \leq a - a_n < \frac{1}{n}$. Now the result follows by applying the squeeze rule ■

Exercise 4.26
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $|a_m - a_n| \leq \frac{1}{|m-n|}$ for all $m \neq n$. Show that $a_1 = a_2 = a_3 = \cdots$.

Solution.
Fix $m \in \mathbb{N}$. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \frac{1}{m-n} = 0$, we can find a positive integer N such that $\frac{1}{|m-n|} < \epsilon$ for all $n \geq N$. Thus, for $n \geq N$ we have $|a_n - a_m| < \epsilon$. This shows that $\lim_{n\to\infty} a_n = a_m$. Since the limit of a sequence is unique and the sequence converges to each of its term, we must have that all the terms are equal ■