NAHM'S EQUATIONS AND ROOT SYSTEMS

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Abstract A method of deriving solutions to Nahm's equations based on root structure of simple Lie algebras is given. As an illustration of this method the recently found solutions to Nahm's equations with tetrahedral and octahedral symmetries are shown to correspond to A_2 and A_3 root systems.

1. It is well known [7] [4] that the N-monopole solutions of the self-dual $SU(2)$ Yang-Mills theory are equivalent to the Nahm data. The latter consist of meromorphic functions T_1, T_2, T_3 defined on the interval $[0,2]$, regular on $(0,2)$ and with values in $N \times N$ matrices. The T_i satisfy Nahm's equations

$$\frac{dT_k(s)}{ds} = \frac{1}{2}\epsilon_{ijk}[T_i(s), T_j(s)], \tag{1}$$

and have simple poles at 0 and 2, the residues of which form an irreducible N-dimensional representation of $su(2)$. Furthermore it is required that the T_i satisfy the reality conditions which, in a suitable basis, read: $T_i^\dagger(s) = -T_i(s)$ and $T_i(s) = T_i^t(2-s)$.

Recently, solutions to the Nahm equations with tetrahedral, octahedral and dodecahedral symmetries were found in [5] [6]. These solutions were derived by requiring that the monopole spectral curve defined by the equation

$$P(\eta, \zeta) \equiv \det(\eta + i(T_1 + iT_2) - 2iT_3\zeta - i(T_1 - iT_2)\zeta^2) = 0$$

has the regular solid symmetry. On the other hand it has been known for some time [8] [9] that Nahm's equations are closely related to the classical Yang-Baxter equation which plays an important role in the theory of integrable models. Solutions to the classical Yang-Baxter equation can be classified using the structure of root spaces of simple Lie algebras [1]. More recently another class of equations resembling Nahm's equations appeared in connection with integrable models of

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Calogero type. These are integrability relations of the elliptic Dunkl differential operators [3]. Classification of solutions to these equations given in [2] involves Weyl groups of classical Lie algebras or, more generally, finite Coxeter groups. All these recent developments in understanding equations appearing naturally in the theory of integrable models strongly suggests that the proper approach for solving Nahm's equations should involve root systems of simple Lie algebras. In this paper we propose an ansatz for solving Nahm's equations based on the root systems of A_n type and we show that the tetrahedral and octahedral solutions of [5] fit into the scheme. Starting with a few assumptions we derive their basic consequences concentrating on the Lie-algebraic interpretation of existing solutions. New solutions to Nahm's equations together with detailed proofs will be presented elsewhere.

2. Recall that any simple Lie algebra $\mathcal{L}$ of rank r corresponding to the root system R is generated by H_μ, $\mu = 1, \ldots, r$ and E_α, $\alpha \in R$ which satisfy the following relations

$$[H_\mu, H_\nu] = 0, \qquad [H_\mu, E_\alpha] = \alpha_\mu E_\alpha,$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \ \ (\text{if } \alpha+\beta \in R), \quad [E_\alpha, E_{-\alpha}] = \sum_{\mu=1}^{r} \alpha_\mu H_\mu,$$

where $\alpha_\mu, N_{\alpha,\beta}$ are complex numbers. Since we are interested in solutions of a matrix equation we take any N-dimensional representation of $\mathcal{L}$ chosen so that $E_\alpha^\dagger = E_{-\alpha}$ and $H_\mu^\dagger = H_\mu$. It is useful to represent Nahm's data as a three-component vector field $\mathbf{T}(s) = (T_1(s), T_2(s), T_3(s))$. Then Nahm's equations take the form $\frac{d}{ds}\mathbf{T}(s) = \mathbf{T}(s) \wedge \mathbf{T}(s)$. We search for solutions of this equation in the form

$$\mathbf{T}(s) = \sum_{\alpha \in R_+} (\mathbf{e}_\alpha(s) E_\alpha + \mathbf{e}_{-\alpha}(s) E_{-\alpha}), \tag{2}$$

where R_+ is a set of positive roots and $\mathbf{e}_\alpha(s)$ are three-dimensional vector fields. The reality condition for the Nahm's data $\mathbf{T}$ imply that $\mathbf{e}_{-\alpha}(s) = -\mathbf{e}^*_\alpha(s)$ while (1) becomes

$$\frac{d\mathbf{e}_\beta(s)}{ds} = \frac{1}{2} \sum_{\alpha \in R} N_{\alpha,\beta-\alpha} \mathbf{e}_\alpha(s) \wedge \mathbf{e}_{\beta-\alpha}(s) \tag{3}$$

with a constraint

$$\sum_{\alpha \in R_+} \alpha_\mu \mathbf{e}_\alpha(s) \wedge \mathbf{e}_{-\alpha}(s) = 0, \qquad \mu = 1, \ldots, r. \tag{4}$$

We are looking for the most symmetric configuration of monopoles, i.e., we require that any exchange of at least two monopoles does not change the configuration of the system. Motivated by the examples discussed in section 3, we find that the maximal symmetry requirement can be expressed as two conditions satisfied for all positive roots α

$$\mathbf{e}_\alpha(s) \cdot \mathbf{e}^*_\alpha(s) = f(s) \tag{5}$$

and

$$\mathbf{e}_\alpha(s) \wedge \mathbf{e}^*_\alpha(s) = ig(s)\mathbf{v}_\alpha \tag{6}$$

where $f(s)$ and $g(s)$ are real functions. We assume that the function $g(s)$ does not depend on the phases $\phi_{\alpha i}$, $i = 1, 2, 3$, of the components of $\mathbf{e}_\alpha(s)$. Equations (6) combined with (4) lead to the following condition for the vectors $\mathbf{v}_\alpha$,

$$\sum_{\alpha \in R_+} \alpha_\mu \mathbf{v}_\alpha = 0, \qquad \mu = 1, 2, \ldots, r. \tag{7}$$

Since the vectors $\mathbf{v}_\alpha$ are not linearly independent we impose the following irreducibility and normalisation condition. The vector $\mathbf{v}_\alpha$ has a norm 1 if there are no vectors parallel to $\mathbf{v}_\alpha$ among all the other $\mathbf{v}_\beta$. Otherwise we set $\mathbf{v}_\alpha$ to zero. Furthermore if $\mathbf{v}_\alpha = 0$ for all α in a subset P_+ of R_+ and for $\beta \in R_+ - P_+$, $\mathbf{e}_\beta(s) \wedge \mathbf{e}^*_\beta(s) \neq 0$, then $\mathbf{e}_\alpha(s) = 0$ for $\alpha \in P_+$. This last condition prevents a reduction of the solution to Nahm's equations corresponding to a root system of a Lie algebra of rank r to the solution corresponding to a Lie algebra of lower rank.

Combining (5) and (6) one obtains

$$\mathbf{e}_\alpha(s) \cdot \mathbf{e}_\alpha(s) = |f(s)^2 - g^2(s)|^{1/2} e^{i\theta_\alpha(s)},$$

for some functions θ_α. On the other hand, taking the derivative of (5) and using (3) one arrives at the following constraints

$$\sum_{\alpha \in R_+} N_{\alpha,\beta-\alpha}(\mathbf{e}_\alpha(s) \wedge \mathbf{e}_{\beta-\alpha}(s)) \cdot \mathbf{e}^*_{\alpha+\beta}(s) + c.c = \frac{df(s)}{ds} \tag{8}$$

In the A_2 case, this constraint is fulfilled automatically. In the A_3 case, the constraints (8) are solved by imposing that $N_{\alpha,\beta}(\mathbf{e}_\alpha(s) \wedge \mathbf{e}_\beta(s)) \cdot \mathbf{e}^*_{\alpha+\beta}(s)$ are real and equal to each other for all $\alpha, \beta \in R$. This gives

$$\frac{df(s)}{ds} = 4N_{\alpha,\beta}(\mathbf{e}_\alpha(s) \wedge \mathbf{e}_\beta(s)) \cdot \mathbf{e}^*_{\alpha+\beta}(s), \tag{9}$$

3. We now use the above ansatz to derive solutions to Nahm's equations in the A_2 and A_3 cases. In the A_2 case equations (3) read

$$\frac{d\mathbf{e}_{\alpha_1}(s)}{ds} = -\mathbf{e}^*_{\alpha_2}(s) \wedge \mathbf{e}_{\alpha_3}(s), \quad \frac{d\mathbf{e}_{\alpha_2}(s)}{ds} = \mathbf{e}^*_{\alpha_1}(s) \wedge \mathbf{e}_{\alpha_3}(s),$$
$$\frac{d\mathbf{e}_{\alpha_3}(s)}{ds} = -\mathbf{e}_{\alpha_1}(s) \wedge \mathbf{e}_{\alpha_2}(s), \tag{10}$$

where α_1, α_2 are simple positive roots and $\alpha_3 = \alpha_1 + \alpha_2$. The irreducibility condition together with (7) imply that $\mathbf{v}_{\alpha_1} = \mathbf{v}_{\alpha_2} = \mathbf{v}_{\alpha_3} = 0$. This in turn determines the choice of the components of $\mathbf{e}_{\alpha_i}(s)$ to be $e_{\alpha_i j} = e_j \delta_{ij}$ and reduces equations (10) to

$$\frac{de_1(s)}{ds} = -e^*_2(s)e_3(s), \quad \frac{de_2(s)}{ds} = -e^*_1(s)e_3(s), \quad \frac{de_3(s)}{ds} = -e_1(s)e_2(s). \tag{11}$$

The system of equations (11) can easily be solved using the technique described in [2]. One finds

$$f(s) = \mathcal{P}(u), \qquad u = \zeta^{1/3} s + s_0,$$

where ζ and s_0 are constants and $\mathcal{P}$ is the Weierstrass elliptic function given as a solution of the equation $\mathcal{P}'(u)^2 = 4(\mathcal{P}(u)^3 - 1)$. Setting $e_i(s) = f(s)^{1/2} e^{i\theta_i(s)}$ one finds that $\theta_2 = \theta_1 + c_1$, $\theta_3 = -\theta_1 + c_2$, and

$$\tan\left(\frac{1}{2}\left(3\theta_1(s) + c_1 - c_2\right)\right) = \pm c_3 \exp(3\int f(s)^{1/2} ds), \tag{12}$$

where c_1, c_2, c_3 are constants of integration. From the point of view of the monopole dynamics a spectral curve is the gauge invariant object, i.e., different Nahm's data that correspond to the same spectral curve describe the same monopole. It can be easily checked that the spectral curve in the A_2 case does not depend on the functions θ_i. Therefore there is a freedom of fixing constants c_i and the function θ_1 restricted by (12). It is convenient to choose $c_1 = c_2 = 0$ and $\tan(\theta_1(s)) = 2/\mathcal{P}'(u)$. This choice leads to

$$e_1(s) = e_2(s) = \frac{\mathcal{P}'(u)}{2\mathcal{P}(u)} + i\frac{1}{\mathcal{P}(u)}, \qquad e_3(s) = -e_1^*(s). \tag{13}$$

In the three-dimensional representation of A_2 given by

$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\alpha_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the Nahm data read

$$\mathbf{T} = \left(\begin{pmatrix} 0 & e_1 & 0 \\ -e_1^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -e_1^* \\ 0 & 0 & 0 \\ e_1 & 0 & 0 \end{pmatrix} \right).$$

With e_1 given by (13) this is precisely the solution to Nahm's equations with tetrahedral symmetry found in [5].

As a second example we take the root system of A_3. There are six positive roots α_i, $i = 1, \ldots 6$ which we choose so that $\alpha_1, \alpha_2, \alpha_3$ are simple roots and $\alpha_4 = \alpha_1 + \alpha_2$, $\alpha_5 = \alpha_2 + \alpha_3$, and $\alpha_6 = \alpha_1 + \alpha_2 + \alpha_3$. Nahm's equations (3) read now

$$\begin{aligned} \mathbf{e}'_{\alpha_1} &= -\mathbf{e}_{\alpha_4} \wedge \mathbf{e}^*_{\alpha_2} - \mathbf{e}_{\alpha_6} \wedge \mathbf{e}^*_{\alpha_5}, \qquad & \mathbf{e}'_{\alpha_2} &= \mathbf{e}_{\alpha_4} \wedge \mathbf{e}^*_{\alpha_1} - \mathbf{e}_{\alpha_5} \wedge \mathbf{e}^*_{\alpha_3}, \\ \mathbf{e}'_{\alpha_3} &= \mathbf{e}_{\alpha_5} \wedge \mathbf{e}_{\alpha_2} + \mathbf{e}_{\alpha_6} \wedge \mathbf{e}^*_{\alpha_4}, \qquad & \mathbf{e}'_{\alpha_4} &= \mathbf{e}_{\alpha_1} \wedge \mathbf{e}_{\alpha_2} - \mathbf{e}_{\alpha_6} \wedge \mathbf{e}^*_{\alpha_3}, \\ \mathbf{e}'_{\alpha_5} &= \mathbf{e}_{\alpha_2} \wedge \mathbf{e}_{\alpha_3} + \mathbf{e}_{\alpha_6} \wedge \mathbf{e}^*_{\alpha_1}, \qquad & \mathbf{e}'_{\alpha_6} &= \mathbf{e}_{\alpha_1} \wedge \mathbf{e}_{\alpha_5} - \mathbf{e}_{\alpha_3} \wedge \mathbf{e}_{\alpha_4}. \end{aligned} \tag{14}$$

The irreducibility conditions and the constraints (7) can be easily solved to give

$$\mathbf{v}_{\alpha_1} = \frac{1}{\sqrt{2}}(0,1,1), \qquad \mathbf{v}_{\alpha_2} = \frac{1}{\sqrt{2}}(1,0,1), \qquad \mathbf{v}_{\alpha_3} = \frac{1}{\sqrt{2}}(0,-1,1),$$

$$\mathbf{v}_{\alpha_4} = \frac{1}{\sqrt{2}}(-1,-1,0), \qquad \mathbf{v}_{\alpha_5} = \frac{1}{\sqrt{2}}(-1,1,0), \qquad \mathbf{v}_{\alpha_6} = \frac{1}{\sqrt{2}}(1,0,-1).$$

These then are reflected by the relations between the components of vector fields $\mathbf{e}_{\alpha_i}$, namely $e_{\alpha_1 2} = -e_{\alpha_1 3}$, $e_{\alpha_2 1} = -e_{\alpha_2 3}$, $e_{\alpha_3 2} = e_{\alpha_3 3}$, $e_{\alpha_4 1} = -e_{\alpha_4 2}$, $e_{\alpha_5 1} = e_{\alpha_5 2}$, $e_{\alpha_6 1} = e_{\alpha_6 3}$. Therefore there are at most two different moduli of the components in each $\mathbf{e}_{\alpha_i}$. Let $h_{\alpha_i,1}$ be a modulus of the component of $\mathbf{e}_{\alpha_i}$ which occurs once and $h_{\alpha_i,2}$ be a modulus of the component of $\mathbf{e}_{\alpha_i}$ which occurs twice. The assumption that $g(s)$ does not depend on the phases of $e_{\alpha j}$ implies that for each α there is $k_\alpha \in \mathbf{Z}$ such that $\phi_{\alpha,i} - \phi_{\alpha,j} = (2k_\alpha + 1)\pi/2$, where $e_{\alpha i} = h_{\alpha,1}\exp(i\phi_{\alpha,i})$ and $e_{\alpha j} = h_{\alpha,2}\exp(i\phi_{\alpha,j})$. Choosing $g(s)$ to be positive one finds

$$f(s) = h_{\alpha,1}(s)^2 + 2h_{\alpha,2}(s)^2, \qquad g(s) = \sqrt{8}h_{\alpha,1}(s)h_{\alpha,2}(s),$$

for all positive roots α. Therefore there exist functions $u(s)$ and $v(s)$, independent of α, such that $h_{\alpha 1} = u$ and $h_{\alpha 2} = v$.

Next we take a closer look at the structure of phases $\phi_{\alpha,i}$. From (9) it follows that $\phi_{\alpha,i} + \phi_{\beta,i} = \phi_{\alpha+\beta,i} + k_{\alpha+\beta}\pi$ and that $k_{\alpha_6} = k_{\alpha_4} - 1$. Furthermore from (14) one deduces that the phases $\phi_{\alpha,i}$ are independent of s. Using the gauge freedom we may choose $k_{\alpha_6} = 0$. This choice reduces the system of equations (14) to the two-dimensional problem

$$\frac{dv}{ds} = 2uv, \qquad \frac{du}{ds} = 2(v^2 - u^2), \tag{15}$$

a solution of which is given by

$$v(s) = c^{1/4}\frac{\mathcal{P}(t) + i}{\mathcal{P}(t) - i}, \qquad u(s) = \frac{1}{2}\frac{d}{ds}\log v(s), \tag{16}$$

where $t = \pm\sqrt{2}c^{1/4}e^{i\pi/4}s$, $c = v^2(v^2 - 2u^2)$ is the integration constant and $\mathcal{P}$ is the Weierstrass elliptic function given by the equation $\mathcal{P}'(t)^2 = \mathcal{P}(t)(\mathcal{P}(t)^2 - 1)$. The solution (16) is equivalent to the octahedral solution to Nahm's equations found in [5].

4. In this brief paper we have described a method of deriving symmetric Nahm's data from the root systems of simple Lie algebras of A_n type. We have shown that the tetrahedral and octahedral monopole configurations correspond to root systems of A_2 and A_3 type. For the sake of brevity and clarity we skipped all the proofs and detailed derivations of the results. We intend to present them in a forthcoming full-size article. We also intend to give new solutions to Nahm's equations as well as proofs of non-existence of solutions for certain root systems (such as A_4 for example). Finally we would like to mention that each solution to Nahm's equation obtained in the way described in this paper gives rise to an integrable model of particles interacting on the line. A detailed description and solutions of these models are currently being investigated.

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