More Approximation

Design and Analysis of Algorithms
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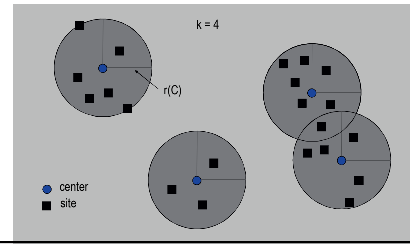
Center Selection Problem

Input. Set of n sites $s_1, \ldots, s_n$.

Center selection problem (informal).

Select k centers C so that maximum distance from a site to nearest center is minimized.



Center Selection Problem

Input. Set of n sites $s_1, \ldots, s_n$.

Notation.

- dist(x, y) = distance between x and y.
- $dist(s_i, C) = \min_{c \in C} dist(s_i, c)$ = distance from s_i to closest center.
- $r(C) = \max_i dist(s_i, C)$ = smallest covering radius.

Goal. Find set of centers C that minimizes r(C), subject to |C| = k.

Distance function properties.

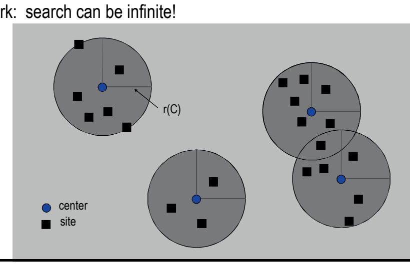
- dist(x, x) = 0 (identity)
- dist(x, y) = dist(y, x) (symmetry)
- $dist(x, y) \le dist(x, z) + dist(z, y)$ (triangle inequality)

Center Selection Problem

Example:

each site is a point in the plane, a center can be any point in the plane, dist(x, y) is the Euclidean distance.

Remark: search can be infinite!

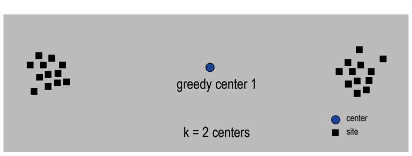


Greedy Algorithm: A False Start

Greedy algorithm:

Put the first center at the best possible location for a single center, and then keep adding centers so as to reduce the covering radius each time by as much as possible.

Remark: arbitrarily bad!



Greedy Algorithm

Greedy algorithm:

Repeatedly choose the next center to be the site farthest from any existing center.

Greedy-Center-Selection(k, n, $s_1, s_2, \ldots, s_n$)

```
set C:=∅
repeat k times
   select a site s_i with maximum dist(s_i, C)
   add s_i to C
endrepeat
return C
```

↑ site farthest from any center

Observation.

Upon termination all centers in C are pairwise at least r(C) apart.

Analysis of Greedy Algorithm

Theorem
Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

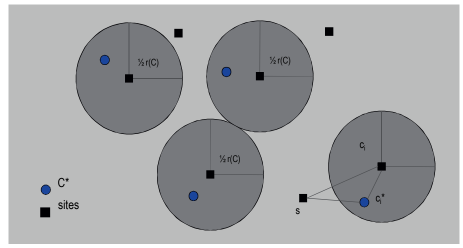
Proof.
By contradiction. Assume $r(C^*) < 1/2\, r(C)$.
- For each site c_i in C, consider ball of radius ½ r(C) around it.
- Exactly one c_i^* in each ball; let c_i be the site paired with c_i^*.
- Consider any site s and its closest center c_i^* in C^*.
- $\mathrm{dist}(s, C) \le \mathrm{dist}(s, c_i) \le \mathrm{dist}(s, c_i^*) + \mathrm{dist}(c_i^*, c_i) \le 2r(C^*)$.
- Thus $r(C) \le 2r(C^*)$.

QED

Δ-inequality

$\le r(C^*)$ since c_i^* is closest center

Analysis of Greedy Algorithm: Picture



Analysis of Greedy Algorithm

Corollary
Greedy algorithm is a 2-approximation algorithm for center selection problem.

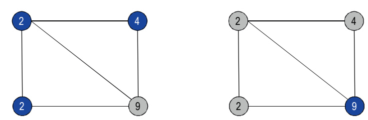
Remark.
Greedy algorithm always places centers at sites, but is still within a factor of 2 of best solution that is allowed to place centers anywhere.

Question. Is there hope of a 3/2-approximation? 4/3?

Theorem
Unless P = NP, there is no ρ-approximation for center-selection problem for any $\rho < 2$.

Weighted Vertex Cover

Weighted vertex cover.
Instance
A graph G with vertex weights
Objective
Find a vertex cover of minimum weight.



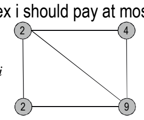
weight = 2 + 2 + 4 weight = 9

Pricing method.

Each edge must be covered by some vertex i. Edge e pays price $p_e \ge 0$ to use vertex i.

Fairness. Edges incident to vertex i should pay at most w_i in total.

for each vertex $i: \sum_{e=(i,j)} p_e \le w_i$



Claim.
For any vertex cover S and any fair prices p_e: $\sum_e p_e \le w(S)$.

Proof.

$$\sum_{e\in E} p_e \le \sum_{i\in S} \sum_{e=(i,j)} p_e \le \sum_{i\in S} w_i = w(S).$$

each edge e covered by at least one node in S

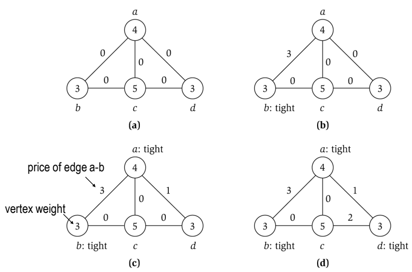
sum fairness inequalities for each node in S

Pricing Method

Pricing method: Set prices and find vertex cover simultaneously.

```
Weighted-Vertex-Cover-Approx(G,w)
  for each e∈E
     set pe:=0
  while there is edge i-j such that neither i nor j
  are tight do
       select such an edge e
       increase pe without violating fairness
  endwhile
  set S:= set of all tight nodes
  return S
```

Pricing Method



Analysis of Pricing Method

Theorem
Pricing method is a 2-approximation algorithm

Proof

Algorithm terminates since at least one new node becomes tight after each iteration of while loop.

Let S = set of all tight nodes upon termination of algorithm.

S is a vertex cover: if some edge i-j is uncovered, then neither i nor j is tight.

But then while loop would not terminate.

Analysis of Pricing Method

Proof (cntd)

Let S* be optimal vertex cover. We show w(S) ≤ 2w(S*).

$$w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e \le \sum_{i \in V} \sum_{e=(i,j)} p_e = 2 \sum_{e \in E} p_e \le 2w(S^*).$$

↑ all nodes in S are tight ↑ S ⊆ V, prices ≥ 0 ↑ each edge counted twice ↑ fairness lemma

QED