CATEGORICAL PRODUCT OF SEMIRING-VALUED GRAPHS AND ITS GROUP

SYNOPSIS

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By

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1 Introduction

1.1 Crisp Graph

The theory of graphs plays a vital role in the area of applications of mathematics. It finds its applications in social network analysis, Birth-CPM analysis, internet networks etc. A graph can be modeled to represent any abstract problem in a simple way by means of diagrams. The theory of graphs obtained its significance from the famous Konigsberg Bridge problem which was settled by Euler, the four-colour conjecture, Knight's tour etc.

In the literature, there are four major categories of graph products

1. Cartesian Product
2. Strong Product
3. Categorical Product
4. Lexicographic Product

Weichsel [38], observed that the direct product of two graphs G and H is connected if and only if G and H are connected and not both are bipartite graphs. Since then many different properties of direct product of graphs have been studied, which include structural results, Hamiltonian properties and the well-Known Hedetniemi's conjecture [16] on chromatic number of direct product of two graphs. For a product graph, solving the problems of determining the independence number and the matching number is economical, since the problem size is much smaller in the factors than in the product so that it can be used as a model for concurrency in multiprocessor system.

The lexicographic product is one such type of product which was introduced as the composition of graphs by Harary [5]. The lexicographic product is also known as graph substitution as, $G \circ H$ can be obtained from G by substituting a copy H_g of H for every vertex g of G and then joining all vertices of H_g with all vertices of $H_{g'}$ if $gg' \in E(G)$.

1.2 Theory of Semiring

The term semiring was explicitly introduced by H.S. Vandiver [29] in the year 1934. However, the term was implicitly used by many mathematicians such as, Dedekind [4], Krull [12], Macaulay[13], while studying the structure of the collection of ideals in a ring and the arithmetic theory on the set of positive rationals.

On the one hand, the theory of semirings will be studied to understand the algebraic and structural properties- An abstraction of the theory of rings. On the other hand, the theory of semirings have wider applications in various fields of mathematics such as social network analysis, Automata theory, Cryptography, Optimization problem, Graph theory and so on. A complete survey on the theory of semirings was given by Jonathan Golan [9] in his monograph. It contains various illustration in different branches of mathematics. One such illustration is the concept of $R-$ valued relations on a non-empty set V, whose elements are called vertices or points. The $R-$ relation g on $V \times V$ to R is defined to be $g(v_1, v_2) \neq 0$ for all pairs $v_1, v_2 \in V$. This relation describes a $R-$ valued graph where R is a semiring, in which edges are assigned a non zero value from the semiring R.

1.3 Semiring Valued Graphs

Motivated by the theory of graphs and the theory of semiring and also by Golan's illustration of $R-$ valued graphs, Chandramouleeswaran and others [21], initiated the study of semiring valued graphs (in short, $S-$ valued graphs). Unlike the definition by Golan, the authors assigned $S-$ values to every vertex of a graph. It is well known that every semiring possesses a canonical pre-order. The authors used this canonical pre-order to compare the $S-$ values of the vertices which are connected by an edge. The minimum among the $S-$ values of the end vertices of an edge is assigned as the $S-$ value of the edge under study. One can observe that given a crisp graph G and a semiring $(S, +, \cdot)$ the $S-$valued graph G^S is not unique. Since then many research works have been carried out such as domination [8], [11] in $S-$valued graphs and colouring of $S-$valued graphs [26], [27].

1.4 The Problem Of Study

Motivated by the theory of product of graphs and the theory of semiring valued graphs, we study in our work, the two particular products of $S-$valued graphs, namely categorical product and lexicographic product. We investigate the notion of isomorphism on $S-$valued graphs, connectivity on $S-$valued graphs and the algebraic structure of the collection of all finite simple $S-$valued graphs with the disjoint union and the categorical (Lexicographic) product of $S-$valued graphs. In particular, we have proved that the collection of all $S-$valued graphs under the disjoint union and the categorical product forms a commutative semiring while they forms a near semiring under the lexicographic product.

The study has been generalized into $S-$valued graphs where the edges are assign $S-$values by adding (multiplying) the $S-$values of the end vertices of the given edge, such graphs are represented as S^+-valued graphs ($S^\bullet-$valued graphs). Further, we extend the categorical product of two $S-$valued graphs by considering their $S-$values from different semirings S_1 and S_2, represented by, $G_1^{S_1} \times G_2^{S_2}$. Such product is known as the generalized categorical product of $S-$valued graphs.

1.5 Outlay Of the Thesis

Our main work is divided into seven chapters.

The first chapter **Preliminaries** recalls the basic definitions and fundamental results that are needed for our sequel. This chapter is divided into three sections. The first section discusses the basic definitions of the theory of graphs and the definitions of graph products and other basic definitions. In the second section, we give the definition of semiring and other basic definitions for our work. The third section discusses the the concept of $S-$valued graphs and the fundamental results.

The chapter **Connectivity of $S-$valued Graphs** discusses the notion of connectivity in the $S-$valued graphs. This chapter is divided into two sections.

In the first section, we discuss the concept of $S-$ vertex connectivity of $S-$ valued graphs and the second section discusses the notion of $S-$ edge connectivity of $S-$ valued graphs.

In the chapter **Categorical Product of $S-$ valued graphs**, we introduce the notion of categorical product of $S-$ valued graphs. This chapter is divided into three sections. In the first section, we discuss the concept of categorical product of $S-$ valued graphs and its properties. The second section discusses the regularity conditions on categorical product of $S-$ valued graphs and in the third section we discuss the irregularity conditions on categorical product of $S-$ valued graphs.

The chapter **Colouring on Categorical product of $S-$ valued graphs** is divided into two sections. The first section, discusses the notion of $S-$ vertex colouring on categorical product of $S-$ valued graphs and in the second section, we discuss the concept of $S-$ edge colouring on categorical product of $S-$ valued graphs.

The chapter **Algebraic structures on Categorical Product of $S-$ valued graphs** is divided into two sections. In the first section, we discuss the notion of Isomorphism of $S-$ valued graphs. The second section, discusses the Algebraic structures of collection of all $S-$ valued graphs under the disjoint union and the categorical product of $S-$ valued graphs.

The chapter **Lexicographic Product of $S-$ valued Graphs** is divided into two sections. In the first section, we introduce the notion of lexicographic product of $S-$ valued graphs and the second section, discusses the algebraic structures of collection of $S-$ valued graphs under disjoint union and the lexicographic product of $S-$ valued graphs.

The chapter **A Generalization of $S-$ valued graphs** is divided into three sections. The first section discusses the generalized categorical product of $S-$ valued graphs and in the second section, we discuss the notion of $S^{+}-$ valued graphs. The third section, discusses the concept of $S^{\bullet}-$ valued graphs.

Our thesis ends with a detailed **Bibliography**

2 Preliminaries

In this chapter, we recall basic definitions and fundamental results that are required for our work.

- A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a non empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v; the vertices u and v are called the ends of e.

- A graph $G = (V, E, \psi_G)$ is said to be finite if both its vertex set V and edge set E are finite. Empty graph can be defined as a graph in which $V = \phi$ and $E = \phi$. A graph without any edges is called a Null graph. Every vertex in a null graph is an isolated vertex.

- The Categorical product of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$, and for which vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$.

  Thus $V(G \times H) = \{(g, h) \mid g \in V(G) \textit{ and } h \in V(H)\}$,

  $E(G \times H) = \{(g, h)(g', h') \mid gg' \in E(G) \textit{ and } hh' \in E(H)\}$.

- The Lexicographic product of the graphs G and H is the graph, $G \circ H$, whose vertex set is $V(G) \times V(H)$, and for which $(g, h)(g', h')$ is an edge of $G \circ H$ if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$.

- A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\cdot$ such that

  1. $(S, +, 0)$ is a monoid.
  2. $(S, \cdot)$ is a semigroup.
  3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
  4. $0 \cdot x = x \cdot 0 = 0 \; \forall \; x \in S$.

- Let $(S, +, \cdot)$ be a semiring. A relation $\preceq$ is said to be a Canonical pre-order if for $a, b \in S$, $a \preceq b$ if and only if, there exists $c \in S$ such that $a + c = b$.

- To compare the elements of $S \times \mathbb{N}$, we define $\preceq$ as follows: For all $s_1, s_2 \in S$ and $n, m \in \mathbb{N}$

  1. $(s_1, n) \preceq (s_2, m) \Leftrightarrow s_1 \preceq s_2$ and $n \leq m$.
  2. if $s_1 \preceq s_2$ and $n \geq m$, the comparision is with respect to the $S-$ values.

- Let $(S, +, \cdot)$ and $(S', +, \cdot)$ be two given semirings. A map $\beta : S \to S'$ is said to be a semiring homomorphism if $\beta(a + b) = \beta(a) + \beta(b)$ and $\beta(a \cdot b) = \beta(a) \cdot \beta(b)$ for all $a, b \in S$. If $(S, +)$ and $(S', +)$ are monoids, then the homomorphism $\beta : S \to S'$ satisfies $\beta(0_S) = 0_{S'}$. If $(S, \cdot)$ and $(S', \cdot)$ are also monoids then the homomorphism $\beta : S \to S'$ satisfies $\beta(1_S) = 1_{S'}$.

- An algebraic structure $(S, +, \cdot)$ is said to be right near-semiring with a constant 0 if it satisfies the following axioms:

  1. $(S, +, 0)$ is a monoid.
  2. $(S, \cdot)$ is a semigroup.
  3. $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in S$.
  4. $0 \cdot a = 0$ for all $a \in S$.

- Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a semiring valued graph (or a S-valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be $\psi(x, y) = \begin{cases} min\{\sigma(x), \sigma(y)\} & if\ \sigma(x) \preceq \sigma(y)\ (or)\ \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$

  for every unordered pair (x, y) of $E \subset V \times V$. we call σ, a $S-$ vertex set and ψ a $S-$ edge set of $S-$ valued graph G^S.

- A $S-$ valued graph G^S is said to be

  1. vertex regular if $\sigma(v) = a\ \forall\ v \in V$ and for some $a \in S$
  2. Edge regular if $\psi(u, v) = a\ \forall (u, v) \in E$ and for some $a \in S$.
  3. $S-$ regular if it is both vertex as well as edge regular.

- A $S-$ valued graph G^S is said to be degree regular $S-$ valued graph (d_S- regulargraph) if $deg_S(v) = (a, n)\ \forall\ v \in V$ and some $a \in S$ and $n \in \mathbb{Z}_+$.

- Any $S-$ valued grpah G^S which is not $S-$ regular is called a $S-$ valued irregular (simply $S-$ irregular graph).

- A $S-$ valued graph G^S is said to be a weight $S-$ vertex irregular if for every vertex $v \in V, \sigma(v) \neq \sigma(u), u \in N_S(v)$.

- Let $G^S = (V, E, \sigma, \psi)$ be a given graph. Let $v \in V$. Consider the open neighbourhood $N_S(v)$ of v. Let $u_1, u_2 \in N_S(v)$ such that $\sigma(u_1) \neq \sigma(v)$ and $\sigma(u_2) \neq \sigma(v)$. If $\sigma(u_1) \neq \sigma(u_2)$ then, G^S is said to be neighbourly weight $S-$ irregular at v. If G^S is neighbourly weight $S-$ irregular at every vertex v, then G^S is called a neighbourly weight $S-$ irregular graph.

- Consider a $S-$ valued graph $G^S = (V, E, \sigma, \psi)$. Let e_i^j denotes the edge $(v_i, v_j) \in E$. Then the $S-$ edge imbalance of e_i^j is denoted by $Imb_S(e_i^j)$ and is defined as $Imb_S(e_i^j) = \Big(\psi(e_i^j), |d_G(v_i) - d_G(v_j)|\Big)$, where $d_G(v_i)$ denotes the degree of a vertex v_i in crisp graph G.

- Let $G^S = (V, E, \sigma, \psi)$ be a $S-$ valued graph. Then, the $S-$ irregularity of G^S is defined by $Irr_S(G^S) = \sum\limits_{e_i^j \in E(G^S)} Imb_S(e_i^j)$.

- The total $S-$ irregularity of a $S-$ valued graph $G^S = (V, E, \sigma, \psi)$ is defined for all pairs of vertices $(v_i, v_j), i < j \in V(G^S)$, such that

$$Tirr_S(G^S) = \sum_{\substack{(v_i,v_j) \in V(G^S) \\ i<j}} \Big(\min\left\{\sigma(v_i, \sigma(v_j)\right\}, |d_G(v_i) - d_G(v_j)|\Big).$$

- Consider the $S-$ valued graph $G^S = (V, E, \sigma, \psi)$. Let $\mathcal{C} = \{c_1, c_2, \cdots\}$ be a set of colours . A colouring of G^S is given by a function $f : V \times V \to S \times \mathcal{C}$ such that for all $v \in V$, $f(u, v) = (\sigma(v), \c{(}v)), c(v) \in \mathcal{C}$.

- Let $G^S = (V, E, \sigma, \psi)$ be a $S-$ valued graph. The vertex chromatic number of G^S denoted by $\chi_S(G^S)$, is defined to be $\chi_S(G^S) = \Big(\underset{v \in V}{min}\sigma(v), min|\mathcal{C}|\Big)$.

Throughout our work we are considering only finite and simple $S-$ valued graphs.

3 Connectivity of $S-$Valued Graphs

In this chapter, we introduce the notion of connectivity of $S-$ valued graphs. The theory of connectivity in graphs plays a crucial role in the study of network problems. It discusses the minimum number of links, whose removal collapses the entire system. In this chapter, analogous to the theory of crisp graphs, we introduce the two types of connectivity, namely, the vertex $S-$ connectivity and the edge $S-$ connectivity. This chapter is divided into two sections. While the first section discusses the notion of vertex $S-$ connectivity, the second section discusses the notion edge $S-$ connectivity.

In this chapter, we have defined the following terms.

- Consider the $S-$ valued graph $G^S = (V, E, \sigma, \psi)$. Let $\mathcal{B}_{G^S} = \{B_i \mid i = 1, \cdots, k\}$ be the collection of $S-$ connected components of G^S. Then $B_i = (P_i, F_i)$, $P_i \subseteq V, F_i \subseteq E, i = 1, \cdots, k$. Therefore $|\mathcal{B}_{G^S}| = k$ and the graph G^S is said to have k connected components. If $k = 1$, then G^S is said to be a $S-$ connected graph in which every pair of vertices have a $S-$ path.

- Let $G^S = (V, E, \sigma, \psi)$ be a $S-$ valued graph with vertex set $V = \{v_i \mid i = 1, \cdots, n\}$ and edge set $E = \left\{(v_i v_j) = e_i^j \mid i, j = 1, \cdots, n\right\}$. Then the vertex strength of the $S-$ valued graph G^S is the sum of the $S-$ values of vertices of G^S. That is, $St_V(G^S) = \sum\limits_{v_i \in V} \sigma(v_i) = \mid V \mid_S$.

- A vertex separating set of a given $S-$ valued graph G^S is a subset $P \subseteq V$ whose removal from G^S reduces the vertex strength of the graph G^S and splits the graph into components. That is $St_V(G^S - P) \preceq St_V(G^S)$ and $|\mathcal{B}_{G^S-P}| > |\mathcal{B}_{G^S}|$. In other words, $|V - P|_S \preceq |V|_S$ and $|\mathcal{B}_{G^S-P}| > |\mathcal{B}_{G^S}|$.

- The vertex $S-$ connectivity of G^S denoted by $\kappa_V^S(G^S)$ is defined as $\kappa_V^S = \underset{P \subseteq V}{min}\{(|P|_S, |P|)\}$, where $P \subseteq V$ such that $St_V(G^S - P) \preceq St_V(G^S)$, and $|\mathcal{B}_{G^S-P}| > |\mathcal{B}_{G^S}|$.

- Let $G^S = (V,,E,\sigma,\psi)$ be a $S-$valued graph with vertex set $V = \{v_i, \mid 1 \leq i \leq n\}$ and edge set $E = \left\{(v_iv_j) = e_i^j \mid 1 \leq i,j \leq n\right\}$. Then the edge strength of the $S-$valued graph G^S is the sum of the $S-$values of edges of G^S. That is, $St_E(G^S) = \sum\limits_{e_i^j \in E} \psi(e_i^j) =\mid E \mid_S$.

- An edge separating set of a given $S-$valued graph G^S is a subset $F \subseteq E$ whose removal from G^S reduces the edge strength of the graph G^S and increases the number of components in G^S. That is $St_E(G^S - F) \preceq St_E(G^S)$ and $|\mathcal{B}_{G^S-F}| > |\mathcal{B}_{G^S}|$. In other words, $|E-F|_S \preceq |E|_S$ and $|\mathcal{B}_{G^S-F}| > |\mathcal{B}_{G^S}|$.

- The edge $S-$connectivity of G^S, denoted by $\kappa_E^S(G^S)$, is defined as $\kappa_E^S = \underset{F\subseteq E}{min}\{(|F|_S, |F|)\}$, where $F \subseteq E$ such that $St_E(G^S - F) \preceq St_E(G^S)$, and $|\mathcal{B}_{G^S-F}| > |\mathcal{B}_{G^S}|$.

We have proved the following results in this chapter.

- $\kappa_V^S \preceq p_S$, where p_S is the order of the graph G^S.

- Consider a $S-$valued graph $G^S = (V,E,\sigma,\psi)$. Let $H^S = (P,F,\sigma_P,\psi_F)$ be a subgraph of G^S obtained by deleting a vertex v in V.

  Then $St_V(H^S) \preceq St_V(G^S)$.

- $\kappa_V^S(H^S) \preceq \kappa_V^S(G^S)$.

- Let $G^S = (V,E,\sigma,\psi)$ be a $S-$valued graph and $H^S = (P,F,\sigma_P,\psi_F)$ be a subgraph of G^S such that $P \subseteq V$ and $F \subseteq E$. Then $St_V(H^S) \preceq St_V(G^S)$.

- Let $H^S = (P,F,\sigma_P,\psi_F)$ be a subgraph of G^S. Then, $\kappa_V^S(H^S) \preceq \kappa_V^S(G^S)$

- Let $G^S = (V,E,\sigma,\psi)$ be a given $S-$valued graph. If C^S is a clique of G^S, then $St_V(C^S) \preceq St_V(G^S)$

- If $\kappa_V^S(G^S) = (\underset{v_i\in V}{min}\sigma(v_i), 0)$, then G^S is either K_1^S or disconnected.

- If G^S is a complete $S-$valued graph with $n-$vertices, then

  $\kappa_V^S(G^S) = \underset{P\subseteq V}{min}\Big(\sum\limits_{v_i\in P}\sigma(v_i), n-1\Big)$ where P is the vertex separating set of G^S.

- If G^S is a S-path, then $\kappa_V^S(G^S) = \underset{v_i \in V}{min}\Big(\sigma(v_i), 1\Big)$.

- If G^S is a $S-$ cycle, then $\kappa_V^S(G^S) = \underset{p \subseteq V}{min}\Big(\sum_{v_i \in P} \sigma(v_i), 2\Big)$.

- If $K_{m,n}^S$ is a complete bipartite $S-$ valued graph with two bipartition sets V_1 and V_2 such that $|V_1| = m; |V_2| = n$.

  Then $\kappa_V^S(G^S) = min\Big\{(\sum_{v_i \in V_1} \sigma(v_i), m), (\sum_{v_i \in V_2} \sigma(v_i), n)\Big\}$

- For a $S-$ star $K_{1,n}^S$ with pole v, $\kappa_V^S(K_{1,n}^S) = (\sigma(v), 1)$.

- Consider a $S-$ valued graph $G^S = (V, E, \sigma, \psi)$. Let $H^S = (P, F, \sigma_P, \psi_F)$ be a subgraph of G^S such that $H^S = (P, F = (E - \{e\})$ for some $e \in E$. Then $St_E(H^S) \preceq St_E(G^S)$.

- $\kappa_E^S(H^S) \preceq \kappa_E^S(G^S)$

- Let $G^S = (V, E, \sigma, \psi)$ be a $S-$ valued graph and $H^S = (P, F, \sigma_P, \psi_F)$ be a subgraph of G^S such that $P \subseteq V$ and $F \subseteq E$. Then $St_E(H^S) \preceq St_E(G^S)$.

- Let $H^S = (P, F, \sigma_P, \psi_F)$ be a subgraph of G^S. Then, $\kappa_E^S(H^S) \preceq \kappa_E^S(G^S)$.

- Let $G^S = (V, E, \sigma, \psi)$ be a given $S-$ valued graph. If C^S is a clique of G^S, then $St_E(C^S) \preceq St_E(G^S)$.

- Let $G^S \neq K_1^S$ be a disconnected $S-$ valued graph. Then $\kappa_E^S(G^S) = \Big(\underset{e_i^j \in E}{min} \psi(e_i^j), 0\Big)$.

- For any vertex $S-$ regular graph $G^S = (V_S, E_S)$, the inequality

$$\kappa_V^S(G^S) \preceq \kappa_E^S(G^S) \preceq \delta_S(G^S)$$

  holds.

- Homomorphic image of a $S-$ path in G_1^S is a $S-$ path in G_2^S. That is $S-$ valued homomorphism preserves $S-$ paths.

- Homomorphic image of a $S-$ cycle in G_1^S is a $S-$ cycle in G_2^S.

4 Categorical Product of $S-$valued Graphs

In this chapter, we introduce the notion of Categorical product of $S-$ valued graphs also study the connectedness properties of $S-$ valued graphs. This chapter is divided into three sections. The first section deals with the definition of Categorical product of $S-$ valued graphs, illustraed with examples also discusses the connectedness properties of categorical product. The second section discusses the regularity conditions on categorical product of $S-$ valed graphs. The third section discusses the irregularity conditions on categorical product of $S-$ valued graphs.

In this chapter, we have defined the following terms.

- Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ where $V_1 = \{v_i \mid 1 \leq i \leq p_1\}$, $E_1 \subseteq V_1 \times V_1$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ where $V_2 = \{u_j \mid 1 \leq j \leq p_2\}$, $E_2 \subseteq V_2 \times V_2$ be two given $S-$ valued graphs.

  Let $V_1 \times V_2 = \{w_{ij} = (v_i, u_j) \mid 1 \leq i \leq p_1; 1 \leq j \leq p_2\}$; and

  $E = E_1 \times E_2 \subseteq V_1 \times V_2$.

  The Categorical product of two $S-$ valued graphs G_1^S and G_2^S is defined by

  $$G_\times^S = G_1^S \times G_2^S = (V, E, \sigma, \psi)$$

  where $V = V_1 \times V_2 = \{w_{ij} = (v_i, u_j) \mid v_i \in V_1 \ and \ u_j \in V_2\}$.

  The two vertices $w_{ij} = (v_i, u_j), w_{kl} = (v_k, u_l)$ are adjacent if $v_i v_k \in E_1$ and $u_j u_l \in E_2$

  Then $E = \left\{e_{ij}^{kl} = (w_{ij}, w_{kl}) \mid e_i^k = v_i v_k \in E_1 \ and \ e_j^l = u_j u_l \in E_2\right\}$

  The $S-$ valued function $\sigma : V \to S$ is defined by

  $\sigma(v_i, u_j) = \min\{\sigma_1(v_i), \sigma_2(u_j)\}$

  and the $S-$ valued function $\psi : E \to S$ is defined by

  $\psi(w_{ij}) = \min\left\{\psi_1(e_i^k), \psi_2(e_j^l)\right\}$

- Let $G_\times^S = G_1^S \times G_2^S$ be the Categorical product of G_1^S and G_2^S.

  We define the projections $\pi_1 : G_\times^S \to G_1^S$ by

  $\pi_1((v_i, u_j), \sigma(v_i, u_j)) = (v_i, \sigma_1(v_i))$

and $\pi_2 : G_\times^S \to G_2^S$ by $\pi_1((v_i, u_j), \sigma(v_i, u_j)) = (u_j, \sigma_2(u_j))$

Clearly, the projections π_1 and π_2 are $S-$valued homomorphisms.

In this chapter, we have proved the following results.

- Suppose (v, u) and (v', u') are vertices of a Categorical product $G_\times^S = G_1^S \times G_2^S$, and n is an integer for which G_1^S has a $S-$path $P^S(vv')$ of length n and G_2^S has a $S-$path $P^S(uu')$ of length n. Then $G_\times^S = G_1^S \times G_2^S$ has a $S-$path of length n from (v, u) to (v', u').

- Suppose there is a $S-$path of length n from (v, u) to (v', u') in $G_\times^S = G_1^S \times G_2^S$. Then, each factor G_1^S and G_2^S of $G_\times^S$ has a $S-$path of length n from v to v' and u to u' respectively.

- Consider the Categorical product $G_\times^S = G_1^S \times G_2^S$. If $G_\times^S = G_1^S \times G_2^S$ is $S-$connected then the factors G_1^S and G_2^S of $G_\times^S$ are also $S-$connected.

- The Categorical product of two $S-$regular graphs is again a $S-$ regular graph

- The Categorical product of two $S-$valued graphs is $S-$regular if the $S-$value corresponding to the $S-$regular graph is minimum among the $S-$values.

- The Categorical product of two edge regular $S-$valued graphs is an edge regular $S-$ valued graph.

- In the categorical product of $S-$valued graphs, every vertex regular $S-$valued graphs is edge regular $S-$valued graph.

  But the converse need not be true, in general.

- The Categorical product of two degree regular $S-$valued graphs (d_S-regular) is again a degree regular $S-$valued graph.

- Let $(S, +, \cdot)$ be a semiring and $a, b \in S$. If G_1^S is $(a, m)-$regular graph and G_2^S is $(b, n)-$regular graph then the Categorical product $G_\times^S = G_1^S \times G_2^S$ is either $(a, mn)-$regular or $(b, mn)-$regular graph, depending on $a \preceq b$ or $b \preceq a$ respectively.

- The Categorical product of two $S-$ irregular graphs is again a $S-$ irregular graph.
- The categorical product of two neighbourly weight $S-$ irregular graphs need not be a neighbourly weight $S-$ irregular graph.
- Consider the two $S-$ valued graphs G_1^S and G_2^S, then $Irr_S(G_\times^S) \preceq Irr_S(G_1^S) + Irr_S(G_2^S)$, where $G_\times^S = G_1^S \times G_2^S$.
- $Irr_S(G_\times^S) \preceq M(G_1^S)Irr_S(G_1^S) + M(G_2^S)Irr_S(G_2^S)$, where $M(G_1^S) = \sum\limits_{v_i \in V_1} deg_S(v_i)$ and $M(G_2^S) = \sum\limits_{u_j \in V_2} deg_S(u_j)$.
- $Tirr_S(G_\times^S) \preceq 2p_S(G_1^S)q_S(G_1^S)Tirr_S(G_1^S) + 2p_S(G_2^S)q_s(G_2^S)Tirr_S(G_2^S)$

5 Colouring On Categorical Product of $S-$Valued Graphs

In [26], the authors introduced the notion of vertex colouring on $S-$ valued graphs. In [27], they have discussed the notion of edge colouring on $S-$ valued graphs. In this chapter, we discuss the colouring on categorical product of two $S-$ valued graphs. We reformulate the definition for vertex colouring given in [26] and the definition of edge colouring in [27]. This chapter is divided into two sections. The first section discusses the $S-$ vertex colouring of categorical product of $S-$ valued graphs and prove some simple results. In the second section we discuss the $S-$ edge colouring of $S-$ valued graphs and study the categorical product of class-1 and class-2 type of $S-$ valued graphs.

In this chapter, we have defined the following terms.

- Let $G^S = (V_S, E_S)$ be a given $S-$ valued graph. Let $\mathcal{C} = \{c_1, c_2, \cdots\}$ be a given collection of colours. A $S-$ vertex colouring of G^S is defined as a mapping $f : V_S \to S \times \mathcal{C}$ given by $f(v, \sigma(v)) = \big(\sigma(v), C(v)\big)$, where $C : V \to \mathcal{C}$.
- A $S-$ vertex colouring f is said to be an equi-weight proper vertex colouring if $\sigma(u) = \sigma(v)$ and $C(u) \neq C(v)$ for all $uv \in E(G^S)$.

- A $S-$ vertex colouring f is said to be a total proper colouring if $\sigma(u) \neq \sigma(v)$ and $C(u) \neq C(v)$ for all $uv \in E(G^S)$.

- Let G^S be a $S-$ valued graph. The $S-$ vertex chromatic number of G^S denoted by $\chi_S(G^S)$, is defined to be $\chi_S(G^S) = \Big(\underset{v_i \in V}{min} |N_S[v_i]|_S, \chi(G)\Big)$, where $\chi(G)$ is a vertex chromatic number of the underlying crisp graph G.

- A $S-$ valued graph G^S is said to be $(a,k)-$ colourable for some $a \in S$, if for all $v_i \in V$, $\sigma(v_i) = a$ and $\chi(G) \leq k$.

- Let $G^S_\times = (V, E, \sigma, \psi) = (V_S, E_S)$ be the categorical product of two $S-$ valued graphs $G^S_1 = (V_1, E_1, \sigma_1, \psi_1) = (V_{1_S}, E_{1_S})$ and $G^S_2 = (V_2, E_2, \sigma_2, \psi_2) = (V_{2_S}, E_{2_S})$. Consider $f_1 : V_{1_S} \to S \times \mathcal{C}$ by $f_1(v_i, \sigma_1(v_i)) = (\sigma_1(v_i), C(v_i))$ and $f_2 : V_{2_S} \to S \times \mathcal{C}$ given by $f_2(u_j, \sigma_2(u_j)) = \big(\sigma_2(u_j), C(u_j)\big)$ are $S-$ vertex colourings of G^S_1 and G^S_2 respectively. Then, the $S-$ vertex colouring on $G^S_\times$ is a function $f : V_S \to S \times \mathcal{C}$ defined by $f\big(w_{ij}, \sigma(w_{ij})\big) = \big(\sigma(w_{ij}), C(w_{ij})\big)$ where $C(w_{ij}) = C(v_i)$ is a vertex colouring of a graph G^S_1.

- Let $G^S = (V_S, E_S)$ be a $S-$ valued graph. Let $\mathcal{C} = \{c_1, c_2, \cdots\}$ be a collection of different colours. A $S-$ edge colouring of G^S is a function $\digamma : E_S \to S \times \mathcal{C}$ such that $\digamma\big(e^k_i, \psi(e^k_i)\big) = \big(\psi(e^k_i), C(e^k_i)\big)$, where $C : E \to \mathcal{C}$.

- A $S-$ edge colouring of G^S is said to be equi weight proper $S-$ edge colouring, if $\psi(e^k_i) = \psi(e^l_j)$ and $C(e^k_i) \neq C(e^l_j)$ for all adjacent edges $e^k_i, e^l_j \in E(G^S)$.

- A $S-$ edge colouring of G^S is said to be a total proper $S-$ edge colouring, if $\psi(e^k_i) \neq \psi(e^l_j)$ and $C(e^k_i) \neq C(e^l_j)$ for all adjacent edges $e^k_i, e^l_j \in E(G^S)$.

- Let G^S be a $S-$ valued graph. The $S-$ edge chromatic number of G^S, denoted by $\chi'_S(G^S)$, is defined to be $\chi'_S(G^S) = \Big(\underset{e^k_i \in E}{max} |N_S[e^k_i]|_S, \chi'(G)\Big)$, where $\chi'(G)$ is the edge chromatic number of the underlying crisp graph G.

- Consider the categorical product $G^S_\times = (V_S, E_S)$ of two $S-$ valued graphs $G^S_1 = (V_{1_S}, E_{1_S})$ and $G^S_2 = (V_{2_S}, E_{2_S})$. A $S-$ edge colouring of $G^S_\times$ is a function $\digamma : V_S \to S \times \mathcal{C}$ defined by $\digamma\big(e^{kl}_{ij}, \psi(e^{kl}_{ij})\big) = \big(\psi(e^{kl}_{ij}), C(e^{kl}_{ij})\big)$, where C is a function $C : E \to \mathcal{C}$ such that no two adjacent edges have the same colour.

- A $S-$ valued graph G^S is said to be of class-1, if $\chi'_S(G^S) = \Delta_S(G^S)$. And it is said to be of class-2 if $\chi'_S(G^S) = \Delta_S(G^S) + (\psi(e_i^k), 1)$.

We have proved the following results in this chapter.

- Every equi-weight proper vertex colouring of a $S-$ valued graph G^S is a $(a,k)-$ colourable graph for some $a \in S$.
- For any $S-$ valued graph G^S, if $\chi_S(G^S) \preceq (a,k)$, where $a \neq 0 \in S$ is the minimum among the $S-$ values. Then G^S is a $(a,k)-$ colourable graph with $\chi(G) \leq k$.
- The categorical product of two equi-weight proper vertex colouring $S-$ valued graphs is again a equi-weight proper vertex colouring $S-$ valued graph.
- The categorical product of two total proper $S-$ vertex colouring $S-$ valued graphs is again a total proper $S-$ vertex colouring $S-$ valued graph.
- For any two $S-$ valued graphs G_1^S and G_2^S,

$$\chi_S(G_1^S \times G_2^S) \preceq min\{\chi_S(G_1^S), \chi_S(G_2^S)\}$$

- The categorical product of two equi weight proper $S-$ edge colouring $S-$ valued graphs is again a equi weight proper $S-$ edge colouring $S-$ valued graph.
- The categorical product of two total proper $S-$ edge colouring $S-$ valued graphs is again a total proper $S-$ edge colouring $S-$ valued graph.
- Let G_1^S and G_2^S be two $S-$ valued graphs such that atleast one of them is of class-1, then the categorical product $G_\times^S = G_1^S \times G_2^S$ is of class-1.
- The categorical product of two class-2 $S-$ valued graphs need not be a class-2 $S-$ valued graph, which is illustrated with an example.

6 Algebraic Structure On Categorical Product of $S-$valued Graphs

In this chapter, we discuss the isomorphism on $S-$ valued graphs. We have proved that, the collection of all $S-$ valued graphs Γ^S under the categorical product forms a commutative semigroup. This chapter is divided into two sections, the first section discusses the isomorphism of $S-$ valued graphs. In the second section, we have proved that the collection of all $S-$ valued graphs forms a commutative semiring under the disjoint union and the categorical product of $S-$ valued graphs.

In this chapter, we have defined the following terms.

- Let $G_1^{S_1} = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^{S_2} = (V_2, E_2, \sigma_2, \psi_2)$ be two given S_1- valued and S_2- valued graphs. A homomorphism $f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ of S-valued graphs is a pair of homomorphisms $\alpha : V_1 \to V_2$ which is a bijection and $\beta : S_1 \to S_2$ such that $\beta(\sigma_1(v_i)) \preceq \sigma_2(\alpha(v_i))$ and $\beta(\psi_1(v_i v_j) \preceq \psi_2(\alpha(v_i)\alpha(v_j))\ \forall v_i, v_j \in V_1$.

- A weak isomorphism $f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ is a pair of homomorphisms $\alpha : V_1 \to V_2$ which is a bijection and $\beta : S_1 \to S_2$ such that $\beta(\sigma_1(v_i)) = \sigma_2(\alpha(v_i)) \forall v_i \in V_1$. A Co-weak isomorphism $f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ is a pair of homomorphisms $\alpha : V_1 \to V_2$ which is a bijection and $\beta : S_1 \to S_2$ such that $\beta(\psi_1(v_i v_j) = \psi_2(\alpha(v_i)\alpha(v_j)) \forall v_i, v_j \in V_1$.

- An isomorphism $f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ is a pair of isomorphisms $\alpha : V_1 \to V_2, \beta : S_1 \to S_2$ are such that $\beta(\sigma_1(v_i)) = \sigma_2(\alpha(v_i)) \forall v_i \in V_1$ and $\beta(\psi_1(v_i v_j) = \psi_2(\alpha(v_i)\alpha(v_j)) \forall v_i, v_j \in V_1$.
  If such an isomorphism from $G_1^{S_1}$ to $G_2^{S_2}$ exists and if both α and β are onto, then $G_1^{S_1}$ is said to be $S-$ valued isomorphic to $G_2^{S_2}$ and we write it as $G_1^{S_1} \cong_S G_2^{S_2}$.

We have proved the following results in this chapter.

- Every weak isomorphism is a Co-weak isomorphism between $S-$ valued graphs.

- Every Co-weak isomorphism need not be a weak isomorphism.

- Weak isomorphism preserves the order of the $S-$ valued graphs.
- Co-weak isomorphism preserves the size of the $S-$ valued graphs.
- If $f : G_1^{S_1} \rightarrow G_2^{S_2}$ is an isomorphism of $S-$ valued graphs, then it preserves both the order and the size of the $S-$ valued graphs.
- The Categorical product of two $S-$ valued graphs is commutative upto $S-$ isomorphism. That is, $G_1^S \times G_2^S \cong_S G_2^S \times G_1^S$.
- The Categorical product of $S-$ valued graphs satisfies the associativity upto $S-$ isomorphism. That is, $G_1^S \times (G_2^S \times G_3^S) \cong_S (G_1^S \times G_2^S) \times G_3^S$.
- Let Γ^S be the set of all finite simple $S-$ valued graphs. Then $(\Gamma^S, \times)$ is a commutative semigroup.
- The set of all finite simple $S-$ valued graphs forms a commutative monoid under the union of $S-$ valued graphs. That is, $(\Gamma^S, \cup)$ is a commutative monoid.
- The collection Γ^S is a commutative semiring under the union and Categorical product of $S-$ valued graphs. That is, $(\Gamma^S, \cup, \times)$ is a commutative semiring.

7 Lexicographic Product of $S-$valued Graphs

In crisp graph theory there are four types of graph products. Lexicographic product is one such type of product which was introduced as the composition of graphs by Harary [5]. The lexicographic product is also known as graph subsititution as, $G \circ H$ can be obtained from G by subsitituting a copy H_g of H for every vertex g of G and then joining all vertices of H_g with all vertices of $H_{g'}$ if $gg' \in E(G)$.

Motivated by this, in this chapter, we introduce the notion of Lexicographic product of two $S-$ valued graphs. We verify some algebraic properties satisfied by the collection of $S-$ valued graphs under lexicographic product. This chapter is divided into two sections. The first section discusses the notion of Lexicographic product of $S-$ valued graphs and its properties. In the second section, we prove that the set of all finite simple $S-$ valued graphs forms a right near-semiring under lexicographic product and disjoint union of graphs.

In this chapter, we have defined the following terms.

- Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two $S-$valued graphs with $V_1 = \{v_i \mid i = 1, 2, \cdots, n\}$ and $V_2 = \{u_j \mid j = 1, 2, \cdots, m\}$. Then the Lexicographic product of G_1^S and G_2^S is defined as the $S-$valued graph

$$G_\circ^S = G_1^S \circ G_2^S = (V, E, \sigma, \psi)$$

  where $V = V_1 \times V_2 = \{w_{ij} = (v_i, u_j) \mid v_i \in V_1, u_j \in V_2\}$ and the two vertices w_{ij} and w_{kl} are adjacent if $v_i v_k \in E_1$ or $v_i = v_k$ and $u_j u_l \in E_2$.
  Then $E = \left\{e_{ij}^{kl} \mid v_i v_k \in E_1 \; or \; v_i = v_k \; and \; u_j u_l \in E_2\right\} \subseteq E_1 \times E_2$.

  The $S-$valued function $\sigma : V \to S$ is defined by

  $\sigma(w_{ij}) = (\sigma_1 \circ \sigma_2)(w_{ij}) = min\{\sigma_1(v_i), \sigma_2(u_j)\}$ and $\psi : E \to S$ is defined by

$$\psi(e_{ij}^{kl}) = (\psi_1 \circ \psi_2)(e_{ij}^{kl}) = \begin{cases} min\{\psi_1(v_i v_k), min\{\sigma_2(u_j), \sigma_2(u_l)\}\} & if \; v_i v_k \in E_1 \\ min\{min\{\sigma_1(v_i), \sigma_1(v_k)\}, \psi_2(u_j u_l)\} & if \; v_i = v_k \; and \; u_j u_l \in E_2 \end{cases}$$

This chapter we have proved the following results.

- The lexicographic product of two $S-$regular graph is again a $S-$ regular graph.

- The lexicographic product of two edge regular $S-$valued graphs is need not be a edge regular $S-$valued graph.

- The lexicographic product of $S-$valued graphs satisfies the associativity upto isomorphism. That is, $G_1^S \circ (G_2^S \circ G_3^S) \cong_S (G_1^S \circ G_2^S) \circ G_3^S$.

- The lexicographic product of $S-$valued graphs satisfies the right distributive law over the union of $S-$valued graphs.

  That is $(G_1^S \cup G_2^S) \circ G_3^S = G_1^S \circ G_3^S \cup G_2^S \circ G_3^S$.

- The lexicographic product of $S-$ valued graphs does not satisfy the left distributive law over the union of $S-$valued graphs.

- The set of all $S-$valued graphs Γ^S forms a near semiring with respect to the operation $\cup$ and $\circ$. That is, $(\Gamma^S, \cup, \circ)$ is a near semiring.

8 A Generalization of $S-$ Valued Graphs

In the previous chapter, we have studied the categorical product of two $S-$ valued graphs G_1^S and G_2^S for a given semiring S. In this chapter, we consider two $S-$ valued graphs $G_1^{S_1}$ and $G_2^{S_2}$, where S_1 and S_2 are two different semirings and we study the categorical product of $G_1^{S_1}$ and $G_2^{S_2}$, called the generalized categorical product of $S-$ valued graphs denoted by $G_\times^{S_1\times S_2}$. This chapter is divided into three sections. The first section discusses the notion of generalized categorical product and prove simple results on regularity conditions of $G_1^{S_1}\times G_2^{S_2}$. The second section discusses the categorical product of S^+- valued graphs and in the third section, we discuss the categorical product of $S^\bullet-$ graphs.

In this chapter, we have defined the following terms.

- Let $G_1^{S_1}=(V_1,E_1,\sigma_1,\psi_1)$ where $V_1=\{v_i \mid i=1,\cdots,p_1\}, E_1\subseteq V_1\times V_1$ and $G_2^{S_2}=(V_2,E_2,\sigma_2,\psi_2)$ where $V_2=\{u_j \mid j=1,\cdots,p_2\}$, $E_2\subseteq V_2\times V_2$ be two different S-valued graphs, S_1 and S_2 are different semirings.

  The Generalized Categorical product of two semiring valued graphs $G_1^{S_1}$ and $G_2^{S_2}$ is defined by

  $$G_\times^{S_1\times S_2}=G_1^{S_1}\times G_2^{S_2}=(V,E,\sigma,\psi)$$

  where $V=V_1\times V_2$

  $$=\{w_{ij}=(v_i,u_j) \mid v_i\in V_1 \; and \; u_j\in V_2, i=1,\cdots,p_1; j=1,\cdots,p_2\}.$$

  The two vertices $w_{ij}=(v_i,u_j), w_{kl}=(v_k,u_l)$ are adjacent if $v_iv_k\in E_1$ and $u_ju_l\in E_2$

  Then $E=E_1\times E_2$

  $=\left\{e_{ij}^{kl}=(w_{ij},w_{kl}) \mid e_i^k=v_iv_k\in E_1 \; and \; e_j^l=u_ju_l\in E_2\right\}\subseteq V\times V.$

  The function $\sigma: V\to S_1\times S_2$ is defined by

  $\sigma\Big(w_{ij}=(v_i,u_j)\Big)=\Big(\sigma_1(v_i),\sigma_2(u_j)\Big)$

  and the function $\psi: E\to S_1\times S_2$ is defined by

  $$\psi\Big(w_{ij}w_{kl}=e_{ij}^{kl}\Big)=\Big(min\{\sigma_1(v_i),\sigma_1(v_k)\}, min\{\sigma_2(u_j),\sigma_2(u_l)\}\Big)$$

  $$=\Big(\psi_1(v_iv_k),\psi_2(u_ju_l)\Big)$$

- Let $G_1^{S^+} = (V_1, E_1, \sigma_1, \psi_1)$ where $V_1 = \{v_i \mid i = 1, 2, \cdots, m\}$ and $G_2^{S^+} = (V_2, E_2, \sigma_2, \psi_2)$ where $V_2 = \{u_j \mid j = 1, 2, \cdots, n\}$ be two given S^+- valued graphs.

  The Categorical product of S^+- valued graphs $G_1^{S^+}$ and $G_2^{S^+}$ is defined by

  $$G_{\times}^{S^+} = G_1^{S^+} \times G_2^{S^+} = (V, E, \sigma, \psi)$$

  where $V = V_1 \times V_2 = \{w_{ij} = (v_i, u_j) \mid v_i \in V_1, u_j \in V_2\}$

  the two vertices (v_i, u_j) and (v_k, u_l) are adjacent if $v_i v_k \in E_1$ and $u_j u_l \in E_2$.

  Then, the edge set

  $E = \left\{e_{ij}^{kl} = (w_{ij}, w_{kl}) \mid e_i^k = v_i v_k \in E_1 \; and \; e_i^j = u_j u_l \in E_2\right\}.$

  Define the $S-$ valued functions, $\sigma : V \to S$ by

  $\sigma(w_{ij}) = \sigma\big((v_i, u_j)\big) = \Big(\sigma_1(v_i) + \sigma_2(u_j)\Big)$

  and $\psi : E \to S$ by $\psi\big(e_{ij}^{kl}\big) = \Big(\psi_1(e_i^k) + \psi_2(e_j^l)\Big)$.

- Let $G_1^{S^\bullet} = (V_1, E_1, \sigma_1, \psi_1)$ where $V_1 = \{v_i \mid i = 1, 2, \cdots, m\}$ and $G_2^{S^\bullet} = (V_2, E_2, \sigma_2, \psi_2)$ where $V_2 = \{u_j \mid j = 1, 2, \cdots, n\}$ be two given $S^\bullet-$ valued graphs.

  The Categorical product of $S^\bullet-$ valued graphs $G_1^{S^\bullet}$ and $G_2^{S^\bullet}$ is defined by

  $$G_{\times}^{S^\bullet} = G_1^{S^\bullet} \times G_2^{S^\bullet} = (V, E, \sigma, \psi)$$

  where $V = V_1 \times V_2 = \{w_{ij} = (v_i, u_j) \mid v_i \in V_1, u_j \in V_2\}$

  the two vertices (v_i, u_j) and (v_k, u_l) are adjacent if $v_i v_k \in E_1$ and $u_j u_l \in E_2$.

  Then, the edge set

  $E = \left\{e_{ij}^{kl} = (w_{ij}, w_{kl}) \mid e_i^k = v_i v_k \in E_1 \; and \; e_i^j = u_j u_l \in E_2\right\}.$

  Define the $S-$ valued functions, $\sigma : V \to S$ by

  $\sigma(w_{ij}) = \sigma\big((v_i, u_j)\big) = \Big(\sigma_1(v_i) \cdot \sigma_2(u_j)\Big)$

  and $\psi : E \to S$ by $\psi\big(e_{ij}^{kl}\big) = \Big(\psi_1(e_i^k) \cdot \psi_2(e_j^l)\Big)$.

We have proved the following results in this chapter.

- The generalized categorical product of two $S-$ regular graphs is again a $S-$ regular graph.
- The generalized categorical product of two $S-$ edge regular $S-$ valued graphs is again a $S-$ edge regular $S-$ valued graph.
- The generalized categorical product of two degree regular $S-$ valued graphs (d_S- regular) is again a degree regular $S-$ valued graph.
- If $G_1^{S_1}$ is $(a, m)-$ regular graph and $G_2^{S_2}$ is $(b, n)-$ regular graph, then the generalized categorical product $G_\times^{S_1 \times S_2} = G_1^{S_1} \times G_2^{S_2}$ is $\Big((a, b), mn\Big)-$ regular $S-$ valued graph where $a \in S_1$ and $b \in S_2$.
- The Categorical product of two S^+- vertex regular graphs is again a S^+- vertex regular graph.
- The Categorical product of two S^+- edge regular valued graphs is a S^+- edge regular graph.
- The Categorical product of two degree regular S^+- valued graph (d_S -regular) is again a degree regular S^+- valued graph.
- The Categorical product of $(a, m)-$ regular and $(b, n)-$ regular S^+- valued graphs is $(a+b, mn)-$ regular S^+- valued graph for some $a, b \in S, m, n \in \mathbb{Z}_+$.
- The Categorical product of two $S^\bullet-$ vertex regular graphs is again $S^\bullet-$ vertex regular.
- The categorical product of two $S^\bullet-$ edge regular graphs is again a $S^\bullet-$ edge regular graph.
- The categorical product of two degree regular $S^\bullet-$ valued graphs is a degree regular $S^\bullet-$ valued graph.
- The categorical product of $(a, m)-$ regular $S^\bullet-$ valued graph and $(b, n)-$ regular $S^\bullet-$ valued graph is a $(ab, mn)-$ regular $S^\bullet-$ valued graph for some $a, b \in S$ and $m, n \in \mathbb{Z}_+$.

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