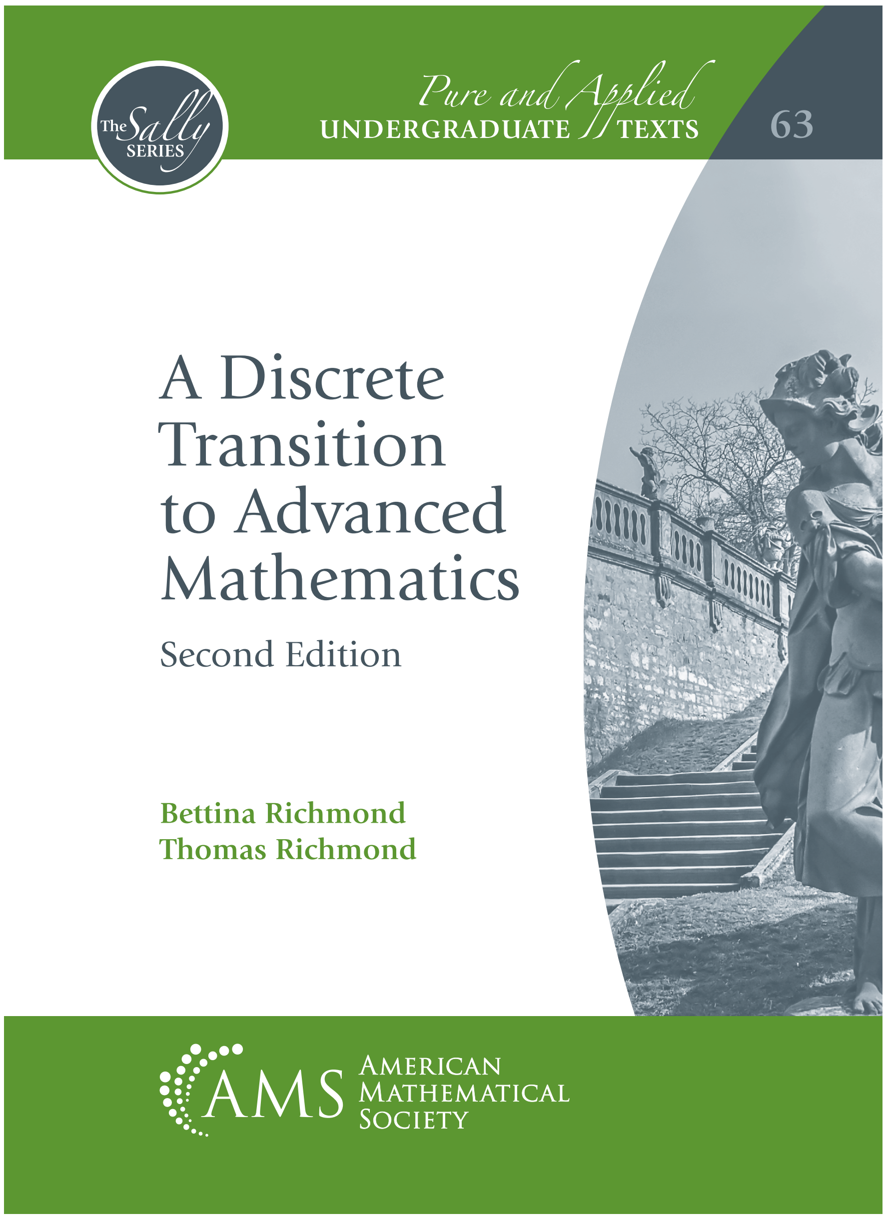
Student Solutions Manual

for



Student Solutions Manual
for

A Discrete Transition to Advanced Mathematics
Second Edition

by

Bettina Richmond and Tom Richmond

Student Solutions Manual prepared by
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This solution manual accompanies *A Discrete Transition to Advanced Mathematics, Second Edition* by Bettina Richmond and Tom Richmond. The text contains over 1015 exercises. This manual includes solutions to parts of 260 of them.

These solutions are presented as an aid to learning the material, and not as a substitute for learning the material. You should attempt to solve each problem on your own and consult the solutions manual only as a last resort.

It is important to note that there are many different ways to solve most of the exercises. Looking up a solution before following through with your own approach to a problem may stifle your creativity. Consulting the solution manual after finding your own solution might reveal a different approach. There is no claim that the solutions presented here are the "best" solutions. These solutions use only techniques which should be familiar to you.

Contents

Chapter 1

Sets and Logic

1.1. Sets

1. (a) True (b) The elements of a set are not ordered, so there is no "first" element of a set.

2. $|\{M, I, S, S, I, S, S, I, P, P, I\}| = |\{M, I, S, P\}| = 4 < 7 = |\{F, L, O, R, I, D, A\}|$.

3. (a) $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$
   (b) $3 \in \{1, 2, 3, 4\}$
   (c) $\{3\} \subseteq \{1, 2, 3, 4\}$
   (d) $\{a\} \in \{\{a\}, \{b\}, \{a, b\}\}$
   (e) $\emptyset \subseteq \{\{a\}, \{b\}, \{a, b\}\}$
   (f) $\{\{a\}, \{b\}\} \subseteq \{\{a\}, \{b\}, \{a, b\}\}$

6. (a) A 0-element set $\emptyset$ has $2^0 = 1$ subset, namely $\emptyset$.

   (b) A 1-element set $\{1\}$ has $2^1 = 2$ subsets, namely $\emptyset$ and $\{1\}$.

   (c) A 2-element set has $2^2 = 4$ subsets.
   A 3-element set has $2^3 = 8$ subsets.
   A 4-element set $\{1, 2, 3, 4\}$ should have $2^4 = 16$ subsets

   (d) The 16 subsets of $\{1, 2, 3, 4\}$ are:

   $$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$$

   $$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$$

   (e) A 5-element set has $2^5 = 32$ subsets.
   A 6-element set has $2^6 = 64$ subsets.
   An n-element set has 2^n subsets.

9. (a) 3, 4, 5, and 7: $|S_3| = |\{t, h, r, e\}| = 4 = |S_4| = |\{f, o, u, r\}| = |S_5| = |\{f, i, v, e\}| = |S_7| = |\{s, e, v, n\}|$.
   (b) $S_{21} = S_{22}$ or $S_{2002} = S_{2000}$, for example.

(c) $a \in S_{1000}$ and $a \notin S_k$ for $k = 1, 2, \ldots, 999$.
(d) (i) True (ii) True (iii) True (iv) False (v) False (vi) True (vii) True (viii) False (ix) True (x) True (xi) True: $\{n, i, e\} = S_9 \in \mathcal{S}$. (xii) True (xiii) False (xiv) True (xv) True (xvi) False

10. (a) $D_1 = \emptyset, D_2 = \{2\}, D_{10} = \{2, 5\}, D_{20} = \{2, 5\}$
(b) (i) True (ii) False (iii) False (iv) True (v) True (vi) False (vii) True (viii) False (ix) True (x) True (xi) False (xii) True
(c) $|D_{10}| = |\{2, 5\}| = 2; |D_{19}| = |\{19\}| = 1$.
(d) Observe that $D_2 = D_4 = D_8 = D_{16}$, $D_6 = D_{12} = D_{18}$, $D_3 = D_9$, $D_{10} = D_{20}$. Thus $|\mathcal{D}| = |\{D_1, D_2, \ldots, D_{20}\}| = |\{D_1, D_2, D_3, D_5, D_6, D_7, D_{10}, D_{11}, D_{13}, D_{14}, D_{15}, D_{17}, D_{19}\}| = 13$.

11. For example, let $S_1 = S_2 = S_3 = \{1, 2, 3\}, S_4 = \{4\}$, and $S_5 = \{5\}$. Now $\mathcal{S} = \{S_k\}_{k=1}^5 = \{\{1, 2, 3\}, \{4\}, \{5\}\}$, so $|\mathcal{S}| = 3$.

1.2. Set Operations

2. (a) $S \cap T = \{1, 3, 5\}$
(b) $S \cup T = \{1, 2, 3, 4, 5, 7, 9\}$
(c) $S \cap V = \{3, 9\}$
(d) $S \cup V = \{1, 3, 5, 6, 7, 9\}$
(e) $(T \cap V) \cup S = \{3\} \cup S = S = \{1, 3, 5, 7, 9\}$
(f) $T \cap (V \cup S) = T \cap \{1, 3, 5, 6, 7, 9\} = \{1, 3, 5\}$.
(g) $V \times T = \{(3,1), (3,2), (3,3), (3,4), (3,5), (6,1), (6,2), (6,3), (6,4), (6,5), (9,1), (9,2), (9,3), (9,4), (9,5)\}$
(h) $U \times (T \cap S) = \{(3,1), (3,3), (3,5), (6,1), (6,3), (6,5), (9,1), (9,3), (9,5)\}$.

4. (a) $A \cap D = \{A\diamondsuit\}$; cardinality 1
(b) $S \cap D = \emptyset$; cardinality 0
(c) $A \cap (S \cup D) = \{A\spadesuit, A\diamondsuit\}$; cardinality 2
(d) $(A \cup K) \cap (S \cup D) = \{A\spadesuit, A\diamondsuit, K\spadesuit, K\diamondsuit\}$; cardinality 4
(e) $(A \cap S) \cup (K \cap D) = \{A\spadesuit, K\diamondsuit\}$; cardinality 2
(f) D^c is the set of all hearts, clubs, and spades; cardinality 39
(g) $K \cap S^c = \{K\clubsuit, K\diamondsuit, K\heartsuit\}$; cardinality 3
(h) $K \cap (S \cup D)^c = \{K\heartsuit, K\clubsuit\}$; cardinality 2
(i) $(A \cup K)^c \cap S = \{2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit\}$; cardinality 11
(j)
$$\begin{aligned} A \times K &= \{(a, k) | \text{ a is an Ace, k is a King}\} \\ &= \{(A\spadesuit, K\spadesuit), (A\spadesuit, K\heartsuit), (A\spadesuit, K\clubsuit), (A\spadesuit, K\diamondsuit), \\ &\quad \{(A\heartsuit, K\spadesuit), (A\heartsuit, K\heartsuit), (A\heartsuit, K\clubsuit), (A\heartsuit, K\diamondsuit), \\ &\quad \{(A\clubsuit, K\spadesuit), (A\clubsuit, K\heartsuit), (A\clubsuit, K\clubsuit), (A\clubsuit, K\diamondsuit), \\ &\quad \{(A\diamondsuit, K\spadesuit), (A\diamondsuit, K\heartsuit), (A\diamondsuit, K\clubsuit), (A\diamondsuit, K\diamondsuit)\}; \end{aligned}$$
cardinality 16
(k) $A \times K^c$ is the set of all ordered pairs of cards of form (a, x) where a is any one of the 4 Aces and x is any one of the 48 cards which are not kings. For example, $(A\heartsuit, A\heartsuit) \in A \times K^c$. Cardinality $4 \times 48 = 192$

(l) $S \times S^c$ is the set of all ordered pairs of cards of form (s, x) where s is any one of the 13 spades and x is any one of the 39 cards which are not spades. Cardinality $13 \times 39 = 507$

(m) $S \setminus K = \{2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, A\spadesuit\}$; cardinality 12

(n) $K \setminus S = \{K\heartsuit, K\clubsuit, K\diamondsuit\}$; cardinality 3

(o) $(K \setminus S) \times S = \{(a, b)|a \in K \setminus S = \{K\heartsuit, K\clubsuit, K\diamondsuit\}$ and $b \in S\}$
$= \{(K\heartsuit, 2\spadesuit), (K\heartsuit, 3\spadesuit), (K\heartsuit, 4\spadesuit), (K\heartsuit, 5\spadesuit), (K\heartsuit, 6\spadesuit), (K\heartsuit, 7\spadesuit), (K\heartsuit, 8\spadesuit),$
$(K\heartsuit, 9\spadesuit), (K\heartsuit, 10\spadesuit), (K\heartsuit, J\spadesuit), (K\heartsuit, Q\spadesuit), (K\heartsuit, K\spadesuit), (K\heartsuit, A\spadesuit), (K\clubsuit, 2\spadesuit),$
$(K\clubsuit, 3\spadesuit), (K\clubsuit, 4\spadesuit), (K\clubsuit, 5\spadesuit), (K\clubsuit, 6\spadesuit), (K\clubsuit, 7\spadesuit), (K\clubsuit, 8\spadesuit), (K\clubsuit, 9\spadesuit),$
$(K\clubsuit, 10\spadesuit), (K\clubsuit, J\spadesuit), (K\clubsuit, Q\spadesuit), (K\clubsuit, K\spadesuit), (K\clubsuit, A\spadesuit), (K\diamondsuit, 2\spadesuit), (K\diamondsuit, 3\spadesuit),$
$(K\diamondsuit, 4\spadesuit), (K\diamondsuit, 5\spadesuit), (K\diamondsuit, 6\spadesuit), (K\diamondsuit, 7\spadesuit), (K\diamondsuit, 8\spadesuit), (K\diamondsuit, 9\spadesuit), (K\diamondsuit, 10\spadesuit),$
$(K\diamondsuit, J\spadesuit), (K\diamondsuit, Q\spadesuit), (K\diamondsuit, K\spadesuit), (K\diamondsuit, A\spadesuit)\}$; cardinality $3 \times 13 = 39$.

11. For each condition given in Exercise 10, give a mutually disjoint collection $\mathcal{C}$ of distinct sets which meets the condition, or explain why no such set is possible.
   (a) $\mathcal{C} = \{(0, 1), (3, 4)\}$.
   (b) Not possible. Any two sets from $\mathcal{C}$ are disjoint, so their intersection is already empty, so $\bigcap \mathcal{C} = \emptyset$ if $\mathcal{C}$ is a mutually disjoint collection.
   (c) $\mathcal{C} = \{[-2, 0], (0, 2]\}$.
   (d) $\mathcal{C} = \{[n, n + 1) : n \in \mathbb{Z}\}$.

15.
$$\begin{aligned}
(x, y) \in A \times (B \cap C) &\iff x \in A, y \in B \cap C \\
&\iff x \in A, y \in B \text{ and } y \in C \\
&\iff x \in A \text{ and } y \in B \text{ and } x \in A \text{ and } y \in C \\
&\iff (x, y) \in (A \times B) \cap (A \times C).
\end{aligned}$$

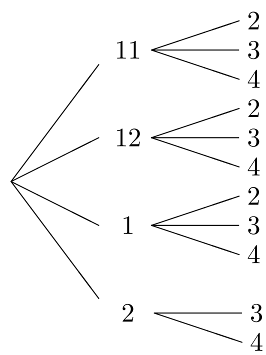
This shows that the elements of $A \times (B \cap C)$ are precisely those of $(A \times B) \cap (A \times C)$, and thus the two sets are equal.

17. The conditions are not equivalent. For example, the collection $\{S_1, S_2\}$ where $S_1 = S_2 \neq \emptyset$ satisfies $(S_i \cap S_j \neq \emptyset \Rightarrow S_i = S_j)$, but not $(S_i \cap S_j \neq \emptyset \Rightarrow i = j)$. However, if the sets of the collection $\{S_i | i \in I\}$ are distinct, the statements will be equivalent.

18. Let A be the set of students taking Algebra and let S be the set of students taking Spanish. Now $|A \cup S| = |A| + |S| - |A \cap S| = 43 + 32 - 7 = 68$. Thus, there are 68 students taking Algebra or Spanish.

20. A tree diagram for the outcomes will have 2 branches for the choice of meat, each stem of which has 7 branches for the possible choices for vegetables, and each of these stems has 5 branches for the choice of dessert. Thus, 2 choices for meat, 7 choices for vegetable, and 5 choices for dessert give $2 \cdot 7 \cdot 5 = 70$ choices for the special.

23. Observe that there are not $4 \cdot 3$ options, for Luis can not take both physics and chemistry at 2:00. There are only 11 scheduling options, as shown in the tree diagram below.



1.3. Partitions

5. (a) Not necessarily. Some B_i may be empty.
   (b) Yes ($S \neq \emptyset$ and $L \neq \emptyset$), $S \cup L = B$, and $S \cap L = \emptyset$.
   (c) No. S and P partition A, but D has nonempty intersection with S or P yet $D \neq S$ and $D \neq P$.
   (d) No. $X = \emptyset$.
   (e) No. $R \cap S = S \neq \emptyset$, but $R \neq S$.

9. (a) Yes.
   (b) No. $L_3 \neq L_4$ even though $L_3 \cap L_4 = \{(0,0)\} \neq \emptyset$. Also, $(0,1) \notin \bigcup \mathcal{D}$.
   (c) Yes.
   (d) Yes.
   (e) No. $(0,1) \notin \bigcup \mathcal{G}$. Also, $P_3 \neq P_4$ yet $P_3 \cap P_4 = \{(0,0)\} \neq \emptyset$.
   (f) No. $(\pi, \pi) \notin \bigcup \mathcal{H}$.

12. Each C_i is nonempty: Given $i \in I$, $B_i \neq \emptyset$, so $\exists b \in B_i$, and $\sqrt{b} \in C_i$.

    $\mathcal{C}$ is a mutually disjoint collection: If $C_i \cap C_j \neq \emptyset$, then $\exists z \in C_i \cap C_j$, and from the definition of C_i and C_j, we have $z^2 \in B_i \cap B_j$. Since $B_i \cap B_j \neq \emptyset$ and $\{B_i | i \in I\}$ is a partition, it follows that $B_i = B_j$, so $\{x \in \mathbb{R} | x^2 \in B_i\} = \{x \in \mathbb{R} | x^2 \in B_j\}$, that is, $C_i = C_j$.

    $\bigcup \mathcal{C} = \mathbb{R}$: Clearly $\bigcup \mathcal{C} \subseteq \mathbb{R}$, so it suffices to show $\mathbb{R} \subseteq \bigcup \mathcal{C}$. Given $x \in \mathbb{R}$, $x^2 \in [0, \infty) = \bigcup \mathcal{B}$, so $x^2 \in B_i$ for some $i \in I$, which shows $x \in C_i$. Thus, $x \in \mathbb{R} \Rightarrow x \in \bigcup \mathcal{C}$, as needed.

15. (a) Given any partition $\mathcal{P}$ of S, each block of $\mathcal{P}$ may be partitioned into singleton sets (that is, into blocks of $\mathcal{D}$), so $\mathcal{D}$ is finer than any partition $\mathcal{P}$ of S.
    (b) The coarsest partition of a set S is the one-block partition $\mathcal{I} = \{S\}$. Given any partition $\mathcal{P}$ of S, the block S of $\mathcal{I}$ is further partitioned by the blocks of $\mathcal{P}$, so ever partition $\mathcal{P}$ of S is finer than $\mathcal{I}$.
    (c) Since each block of the coarser partition $\mathcal{Q}$ is the union of one or more blocks of the finer partition $\mathcal{P}$, we have $|\mathcal{P}| \geq |\mathcal{Q}|$.
    (d) No. $\mathcal{P} = \{(-\infty, 0], (0, \infty)\}$ and $\mathcal{Q} = \{(-\infty, 5], (5, 6), [6, \infty)\}$ are partitions of $\mathbb{R}$ with $|\mathcal{P}| \leq |\mathcal{Q}|$, but neither partition is a refinement of the other.

1.4. Logic and Truth Tables

1. (a) $S\wedge \sim G$ (b) $H\vee \sim S$ (c) $\sim (S \wedge G)$ (d) $(S \wedge G) \vee (\sim H)$
   (e) $(S\vee \sim S) \wedge G$ (f) $S \wedge H \wedge G$ (g) $(S \wedge H) \vee (\sim G)$

7. (a)

P	Q	$P \wedge Q$	$\sim (P \wedge Q)$	$\sim Q$	$\sim (P \wedge Q)\wedge \sim Q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	T	T	T

$\sim (P \wedge Q)\wedge \sim Q = \sim Q$ since the columns for these two statements are identical.

(b) Note that if Q fails, then $(P \wedge Q)$ fails, so that Q fails and $(P \wedge Q)$ fails. On the other hand, if Q fails and some other conditions occur (namely, $(P \wedge Q)$ fails), then Q fails.

10. Answers may vary.

P	Q	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
T	T	T	F	F	F	T	T	T	F	F
T	F	T	T	T	F	T	F	T	F	T
F	T	T	F	F	F	T	T	F	T	T
F	F	F	T	F	T	T	T	T	F	T

(a) $P \vee Q$ (b) $\sim Q$ (c) $P\wedge \sim Q$ (d) $\sim P\wedge \sim Q$ (e) $P\vee \sim P$ (f) $\sim P \vee Q$
(g) $P\vee \sim Q$ (h) $\sim P \wedge Q$ (i) $\sim P\vee \sim Q$

12. The placement of the parentheses in $P \vee Q \wedge R$ is important:
$(P \vee Q) \wedge R \neq P \vee (Q \wedge R)$, as the truth table below indicates.

P	Q	R	$(P \vee Q) \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

14.

P	Q	R	(a)	(b)	(c)	(d)	(e)
T	T	T	T	F	F	T	T
T	T	F	F	F	F	T	T
T	F	T	F	F	F	T	T
T	F	F	F	F	T	F	T
F	T	T	F	F	F	T	T
F	T	F	F	F	F	T	T
F	F	T	F	F	F	T	F
F	F	F	F	T	F	T	T

(a) $P \wedge Q \wedge R$ (b) $\sim P \wedge \sim Q \wedge \sim R$ (c) $P \wedge \sim Q \wedge \sim R$
(d) $\sim (P \wedge \sim Q \wedge \sim R)$ (e) $\sim (\sim P \wedge \sim Q \wedge R)$

1.5. Quantifiers

2. (a) $\forall \epsilon \in (0, \infty)\ \exists n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.
   (b) $\forall e \in \{2k | k \in \mathbb{N} \setminus \{1\}\}\ \exists a \in \{2n | n \in \mathbb{Z}\}$ and $\exists p \in \{\text{prime numbers}\}$ such that $e = ap$.
   (c) $\forall \epsilon \in (0, \infty)\ \exists \delta \in (0, \infty)$ such that $x^2 < \epsilon$ whenever $|x| < \delta$.
   (d) $\exists m \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ with $xy = m$.
   (e) $\forall n \in \mathbb{N} \setminus \{1\}\ \exists p \in \{\text{prime numbers}\}$ such that $n < p < n^2$.
4. Determine whether each statement below is true or false. Give the negation of each statement.
   (a) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}$ such that $y < x$
   True. For $x \in \mathbb{R}$, $y = x - 1 < x$ and $y \in \mathbb{R}$.
   Negation: $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $y \geq x$.
   (b) $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}$ such that $y < x$
   False. $x = 1 \in \mathbb{N}$, but there is no $y \in \mathbb{N}$ with $y < 1$.
   Negation: $\exists x \in \mathbb{N}$ such that $\forall y \in \mathbb{N}$, $y \geq x$.
   (c) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R},\ \ y < x$
   False. For $x \in \mathbb{R}$, $y = x + 1 \not< x$ and $y \in \mathbb{R}$.
   Negation: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}$ such that $y \geq x$.
   (d) $\exists a \in \mathbb{R}$ such that $\forall x \in \mathbb{R},\ \sin a \geq \sin x$
   True. $a = Pi/2 + 2\pi n$ for any $n \in \mathbb{Z}$.
   Negation: $\forall a \in \mathbb{R}$, $\exists x \in \mathbb{R}$ such that $\sin a < \sin x$.
6. (a) True. Take $x = \pm 1$.
   Negation: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $\frac{y}{x} \notin \mathbb{Z}$.
   (b) False. For $a = 0$, $\frac{b}{a}$ is not even defined.
   Negation: $\exists a \in \mathbb{Z}$ such that $\forall b \in \mathbb{Z}$, $\frac{b}{a} \notin \mathbb{Z}$.
   (c) True. $\forall u \in \mathbb{N}$, take $v = 2u$.
   Negation: $\exists u \in \mathbb{N}$ such that $\forall v \in \mathbb{N} \setminus \{u\}$, $\frac{v}{u} \notin \mathbb{N}$.
   (d) False. For $u = 1$, $\frac{1}{v} \notin \mathbb{N}\ \ \forall v \in \mathbb{N}$.
   Negation: $\exists u \in \mathbb{N}$ such that $\forall v \in \mathbb{N} \setminus \{u\}$, $\frac{u}{v} \notin \mathbb{N}$.
   (e) True. $\forall a \in \mathbb{N}$, take $b = a^2$ and $c = a$.
   Negation: $\exists a \in \mathbb{N}$ such that $\forall b, c \in \mathbb{N}, ab \neq c^3$.
11. (a) $\exists a, b \in S$ such that $\forall n \in \mathbb{N}, na \leq b$.
   (b) (i) No. (ii) Yes. (iii) No. (iv) Yes.

1.6. Implications

5. (a) $S \Rightarrow U$ is false if and only if the stock market goes up but unemployment does not go up.
   (b) The converse of $S \Rightarrow U$ is false if and only if unemployment goes up but the stock market does not go up.
   (c) The contrapositive of $\sim I \Rightarrow U$ is false if and only if unemployment does not go up and interest rates do not go down.
7. (a) $x^2 = 4$ only if $x = 2$. False.
   Converse: $x^2 = 4$ if $x = 2$. True.
   (b) If $2x \leq x$, then $x^2 > 0$. False (consider $x = 0$).
   Converse: If $x^2 > 0$, then $2x \leq x$. False.

(c) If 2 is a prime number, then 2^2 is a prime number. False.
Converse: If 2^2 is a prime number, then 2 is a prime number. True.
(d) If x is an integer then $\sqrt{x}$ is an integer. False.
Converse: If $\sqrt{x}$ is an integer, then x is an integer. True.
(e) If every line has a y-intercept, then every line contains infinitely many points. True.
Converse: If every line contains infinitely many points then every line has a y-intercept. False.
(f) A line has undefined slope only if it is vertical. True.
Converse: A line has undefined slope if it is vertical. True.
(g) $x = -5$ only if $x^2 - 25 = 0$. True.
Converse: $x = -5$ if $x^2 - 25 = 0$. False.
(h) x^2 is positive only if x is positive. (Assume $x \in \mathbb{R}$.) False.
Converse: x^2 is positive if x is positive. True.

9. Suppose n is a natural number with units digit d. (Thus, $n = 10m + d$ for some non-negative integers m, d with $d \leq 9$.).
(a) $d \in \{0, 5\}$ is a necessary and sufficient condition on d for n to be divisible by 5.
(b) $d \neq 3$ is a necessary condition on d for n to be divisible by 5 which is not sufficient.
(c) $d = 5$ is a sufficient condition on d for n to be divisible by 5 which is not necessary.

16. (a)

P	Q	$P \Rightarrow Q$	$\sim P \Rightarrow Q$	$\sim P \vee Q$	$P \vee Q$	$\sim(P \Rightarrow \sim Q)$	$P \wedge Q$
T	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	F	F
F	F	T	F	T	F	F	F

(b) (iii) and (v): $(P \Rightarrow Q) = (\sim P \vee Q)$; (iv) and (vi): $(\sim P \Rightarrow Q) = (P \vee Q)$; (vii) and (viii): $\sim(P \Rightarrow \sim Q) = (P \wedge Q)$.

Chapter 2

Proofs

2.1. Proof Techniques

4. (a) Show that if $a, b \in \mathbb{Z}$, then $a\sqrt{2} + b\sqrt{3}$ is either zero or irrational. (Note: $\sqrt{2}, \sqrt{3}$, and $\sqrt{6}$ are irrational.)

Suppose $a, b \in \mathbb{Z}$ and $a\sqrt{2}+b\sqrt{3}$ is rational. Write $a\sqrt{2}+b\sqrt{3} = \frac{m}{n}$ where $m, n \in \mathbb{Z}, n \neq 0$. Now $na\sqrt{2} + nb\sqrt{3} = m$, and squaring both sides of the equation gives $2n^2a^2 + 2n^2ab\sqrt{6}3n^2b^2 = m^2$. If $ab \neq 0$, this gives

$$\sqrt{6} = \frac{m^2 - 2n^2a^2 - 3n^2b^2}{2n^2ab},$$

contrary to $\sqrt{6}$ being irrational. Thus, $ab = 0$, so $a = 0$, $b = 0$, or both. Recalling our assumption that $a\sqrt{2} + b\sqrt{3}$ is rational, $a = 0, b \neq 0$ is not possible since $a\sqrt{2}$ is irrational. Similarly, $b = 0, a \neq 0$ is not possible. Thus, $a = b = 0$, so if $a\sqrt{2} + b\sqrt{3}$ is rational then it is zero. It follows that $a\sqrt{2} + b\sqrt{3}$ is either zero or irrational.

(b) Suppose the circle $C_r = \{(x, y) \in \mathbb{R}^2 : (x - \sqrt{2})^2 + (y - \sqrt{3})^2 = r^2\}$ centered at $(\sqrt{2}, \sqrt{3})$ passes through at most one lattice point $(m, n) \in \mathbb{Z}^2$. Suppose C_r passes through lattice points (m, n) and (j, k). Then

$$(m - \sqrt{2})^2 + (m - \sqrt{3})^2 = (j - \sqrt{2})^2 + (k - \sqrt{3})^2,$$

or

$$m^2 + n^2 + 2 + 3 - 2\sqrt{2}m - 2\sqrt{3}n = j^2 + k^2 + 2 + 3 - 2\sqrt{2}j - 2\sqrt{3}k.$$

$$2\sqrt{2}(j - m) + 2\sqrt{3}(k - n) = j^2 + k^2 - m^2 - n^2 \in \mathbb{Z}$$

$$\sqrt{2}(j - m) + \sqrt{3}(k - n) = \frac{j^2 + k^2 - m^2 - n^2}{2} \in \mathbb{Q},$$

and by (a), $j = m$ and $k = n$. Thus, there is at most one lattice point on C_r.

8. (b) Suppose $x, y \geq 0$ are given. Then

$$\begin{aligned} 0 \leq \lfloor x \rfloor &\leq x \\ \text{and } 0 \leq \lfloor y \rfloor &\leq y. \end{aligned}$$

Multiplying these equations gives

$$\lfloor x \rfloor \lfloor y \rfloor \leq xy,$$

so $\lfloor x \rfloor \lfloor y \rfloor$ is an integer which is $\leq xy$. By definition, $\lfloor xy \rfloor$ is the largest integer which is $\leq xy$, so $\lfloor x \rfloor \lfloor y \rfloor \leq \lfloor xy \rfloor$. This argument holds for any $x, y \in [0, \infty)$.

9. Suppose $p(x) = ax^2 + bx + c$ and $p(1) = p(-1)$. The equation $p(1) = p(-1)$ becomes $a + b + c = a - b + c$, and subtracting $(a + c)$ from both sides gives $b = -b$, so $b = 0$. Thus, $p(x) = ax^2 + c$, so $p(2) = 2^2 a + c = (-2)^2 a + c = p(-2)$.
   Conversely, Suppose $p(x) = ax^2 + bx + c$ and $p(2) = p(-2)$. The equation $p(2) = p(-2)$ becomes $4a + 2b + c = 4a - 2b + c$, and again we find that $b = 0$. Thus, $p(x) = ax^2 + c$, so $p(1) = 1^2 a + c = (-1)^2 a + c = p(-1)$.

12. Note that $n^3 + n = n(n^2 + 1)$. Since n and n^2 have the same parity, n and $n^2 + 1$ have opposite parities (that is, one is even and the other is odd). Since any multiple of an even number is even, it follows that $n(n^2 + 1) = n^3 + n$ is even.

15. (a) Suppose a is a multiple of 3, say $a = 3n$ where $n \in \mathbb{Z}$. Then $a = (n - 1) + n + (n + 1)$, the sum of three consecutive integers. Conversely, suppose $a = k + (k + 1) + (k + 2)$ is the sum of three consecutive integers. Then $a = 3(k + 1)$, so a is a multiple of 3.
    (b) No. The sum $1 + 2 + 3 + 4 = 10$ of four consecutive integers is not a multiple of 4, and 8, a multiple of 4, cannot be written as a sum of four consecutive integers: $0 + 1 + 2 + 3 = 6 < 8 < 10 = 1 + 2 + 3 + 4$.
    (c) The sum of k consecutive integers has form $(n+1)+(n+2)+\cdots+(n+k) = kn + (1 + 2 + \cdots + k)$. Since kn is a multiple of k, the sum will be a multiple of k if and only if $1 + 2 + \cdots + k$ is a multiple of k. Thus,
        a is a multiple of k if and only if a may be written as a sum of k consecutive integers
    is true if and only if $1 + 2 + \cdots + k$ is a multiple of k.
    We will see later that $1 + 2 + \cdots + k$ is the k^{th} triangular number and is given by the formula $\frac{k(k+1)}{2}$. Thus, $1 + 2 + \cdots + k$ is a multiple of k if and only if $\frac{k+1}{2} \in \mathbb{Z}$, that is, if and only if k is odd.

20. Suppose $f(x) = mx + 3$. Show that the following are equivalent.
    (a) $\int_0^1 f(x)dx = 7$
    (b) $m = 8$
    (c) $f(2) = 19$
    (a) $\Rightarrow$ (b): $\int_0^1 f(x)dx = 7 \Rightarrow \frac{mx^2}{2} + 3x \mid_0^1 = 7 \Rightarrow \frac{m}{2} + 3 = 7 \Rightarrow m = 8$.
    (b) $\Rightarrow$ (c): $m = 8 \Rightarrow f(x) = 8x + 3 \Rightarrow f(2) = 16 + 3 = 19$.
    (c) $\Rightarrow$ (a): $f(2) = 19 \Rightarrow m(2) + 3 = 19 \Rightarrow m = 8 \Rightarrow f(x) = 8x + 3 \Rightarrow$ $int_0^1 f(x)dx = int_0^1 (8x + 3)dx = 4x^2 + 3x \mid_0^1 = 7$.

21. Direct proof: For any $k \in \mathbb{Z}$, $x_k = \frac{\pi}{2} + 2\pi k$ is a solution to $\sin x = 1$, so $\sin x = 1$ has infinitely many solutions.

Indirect proof: Suppose to the contrary that $\sin x = 1$ has only finitely many solutions. The solution set is nonempty since $\sin(\frac{\pi}{2}) = 1$. Let x_m be the largest member of the solution set. Now $\sin(x_m + 2\pi) = \sin x_m = 1$, so $x_m + 2\pi$ is an element of the solution set which is larger than x_m, contrary to the choice of x_m as the largest solution. Assuming that there were only finitely many solutions gave a contradiction, so there must be infinitely many solutions.

23. Show that if $\frac{3x+1}{x^2+2}$ is irrational, then x is irrational.

We show the contrapositive: If x is rational, then $\frac{3x+1}{x^2+2}$ is rational. Suppose $x = \frac{m}{n}$ where $m, n \in \mathbb{N}, n \neq 0$. Then $\frac{3x+1}{x^2+2} = \frac{\frac{3m}{n}+1}{\frac{m^2}{n^2}+2} = \frac{(3m+n)n}{m^2+2n^2}$. Now $m, n \in \mathbb{Z}$ implies $(3m+n)n, m^2+2n^2 \in \mathbb{Z}$ and $n \neq 0$ implies $m^2+2n^2 \neq 0$, so we have $\frac{3x+1}{x^2+2} \in \mathbb{Q}$.

40. Suppose k and l are distinct lines that intersect. Suppose A and B are points of intersection of k and l. If $A \neq B$, then the two distinct points A and B determine a unique line, contrary to k and l being distinct lines through A and B. Thus, $A = B$. That is, k and l intersect in a unique point.

42. Moving a knight out and back to his original position on the first move effectively gave the other player the first move in the double move chess game. In initial double move chess, moving a knight out and back does not exchange the roles of first player and second player, since the first player was playing initial double move chess and the second player is left with a different game—one in which only one player gets an initial double move.

2.2. Mathematical Induction

2. (b) For $n = 1$, the statement is $1^3 = \frac{1^2(1+1)^2}{4}$, which is true. Suppose the statement holds for $n = k$, that is, suppose $1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$. We wish to show that the statement holds for $n = k+1$, that is, we wish to show that $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$. Adding $(k+1)^3$ to both sides of the induction hypothesis gives

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \left(\frac{(k+1)^2}{4}\right)(k^2 + 4(k+1)) \\ &= \frac{(k+1)^2(k+2)^2}{4}, \end{aligned}$$

as needed. Now the statement holds for $n = 1$ and for $n = k+1$ whenever it holds for $n = k$, so by mathematical induction, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for every natural number n.

(e) For $n = 1$, the statement is $1^2 = \frac{1(2-1)(2+1)}{3}$, which is true. Suppose the statement holds for $n = k$, that is, suppose $1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 =$

$\frac{k(2k-1)(2k+1)}{3}$. Adding $(2k+1)^2$ to both sides of this equation gives

$$\begin{aligned}
&1^2+3^2+\cdots+(2k-1)^2+(2k+1)^2\\
&\quad= \frac{k(2k-1)(2k+1)}{3}+(2k+1)^2\\
&\quad= \frac{2k+1}{3}(k(2k-1)+3(2k+1))\\
&\quad= \frac{2k+1}{3}(2k^2+5k+3)\\
&\quad= \frac{2k+1}{3}(2k+3)(k+1)\\
&\quad= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3},
\end{aligned}$$

so the statement holds for $n=k+1$. Now the statement holds for $n=1$ and for $n=k+1$ whenever it holds for $n=k$, so by mathematical induction, $1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$ for every natural number n.

7. We wish to show that for any $n\in\mathbb{N}$, there exists $m\in\mathbb{N}$ such that $n^3+(n+1)^3+(n+2)^3=9m$. Taking $n=1$, we find that $1^3+2^3+3^3=36=9m$ where $m=4\in\mathbb{N}$. Now suppose that the statement holds for $n=k$. Then $k^3+(k+1)^3+(k+2)^3=9m$ for some $m\in\mathbb{N}$. We wish to show that $(k+1)^3+(k+2)^3+(k+3)^3=9j$ for some $j\in\mathbb{N}$. But

$$\begin{aligned}
(k+1)^3+(k+2)^3+(k+3)^3 &= k^3+(k+1)^3+(k+2)^3+(k+3)^3-k^3\\
&= 9m+(k+3)^3-k^3\\
&= 9m+k^3+9k^2+27k+27-k^3\\
&= 9m+9(k^2+3k+1)\\
&= 9j \quad \text{where } j=m+k^2+3k+1\in\mathbb{N}.
\end{aligned}$$

Now the statement holds for $n=1$ and for $n=k+1$ whenever it holds for $n=k$, so by mathematical induction, for any $n\in\mathbb{N}$, there exists $m\in\mathbb{N}$ such that $n^3+(n+1)^3+(n+2)^3=9m$.

14. Suppose $\alpha>-1, \alpha\neq 0$. We wish to show $(1+\alpha)^n>1+n\alpha$ for $n\geq 2$. Observe that $\alpha>-1$ guarantees that the powers $(1+\alpha)^n$ are all positive. For $n=2$, the statement is $(1+\alpha)^2>1+2\alpha$, which is true since $(1+\alpha)^2=a+2\alpha+\alpha^2$, and $\alpha^2>0$ for $\alpha\neq 0$. Now suppose $(1+\alpha)^k>1+k\alpha$ for $n=k\geq 2$. We wish to show $(1+\alpha)^{k+1}>1+(k+1)\alpha$. But

$$\begin{aligned}
(1+\alpha)^{k+1} &= (1+\alpha)^k(1+\alpha)\\
&> (1+k\alpha)(1+\alpha) \text{ (Induction hypothesis)}\\
&= 1+k\alpha+\alpha+k\alpha^2\\
&= 1+(k+1)\alpha+k\alpha^2\\
&> 1+(k+1)\alpha \text{ since } k\alpha^2>0 \text{ for } \alpha\neq 0.
\end{aligned}$$

Now the statement holds for $n=2$ and for $n=k+1$ whenever it holds for $n=k$, so by mathematical induction, $(1+\alpha)^n>1+n\alpha$ for every natural number $n\geq 2$.

19. (a) $(a_0, a_1, a_2, a_3, a_4, a_5) = (0, 1, 4, 9, 16, 25)$.
    (b) We conjecture that $a_n = n^2 \ \forall n \in \mathbb{N} \cup \{0\}$.
    (c) From (a), the conjecture holds for $n = 0, 1, 2, 3, 4, 5$. Now suppose the conjecture holds for $n = 0, 1, 2, \ldots, k$ where $k \geq 2$. Then, in particular, $a_{k-1} = (k-1)^2$ and $a_k = k^2$. Now

$$\begin{aligned} a_{k+1} &= 2 - a_{k-1} + 2a_k = 2 - (k-1)^2 + 2k^2 \\ &= 2 - (k^2 - 2k + 1) + 2k^2 = k^2 + 2k + 1 \\ &= (k+1)^2, \end{aligned}$$

    so the conjecture holds for $n = k + 1$ whenever it holds for $n = k$ and $n = k - 1$. By the strong form of mathematical induction, $a_n = n^2$ for any integer $n \geq 0$.

26. (b) Any combination of m 4-cent stamps and n 10-cent stamps gives $(4m + 10n)$-cents postage. Since $4m + 10n$ is always even, 4-cent and 10-cent stamps can never be combined to give any odd amount.

27. Assuming that all horses of any n-element set have the same color, the induction step argues that all horses of an $n+1$-element set $H = \{h_1, \ldots, h_{n+1}\}$ have the same color since all horses of the n-element set $H \setminus \{h_1\}$ have the came color C, all horses of the n-element set $H \setminus \{h_{n+1}\}$ have the came color D, and C = D since $H \setminus \{h_1\} \cap H \setminus \{h_{n+1}\} \neq \emptyset$. However, $H \setminus \{h_1\} \cap H \setminus \{h_{n+1}\} = \emptyset$ if $n = 1$. Thus, the first induction step (if true for $n = 1$, then true for $n = 2$) fails.

2.3. The Pigeonhole Principle

5. (a) 24. Worst case: First 8 nickels, 10 dimes, 3 quarters, then 3 pennies.
   (b) 9. The pigeonhole principle applies. Worst case: 2 of each of the 4 types, then one more.
   (c) All 33. Worst case: the last coin drawn is a quarter.
   (d) 25. Worst case: First 12 pennies, 8 nickels, 3 quarters, then 2 dimes.
   (e) 16. Worst case: First 12 pennies, then 4 more coins to get a second pair.

9. Given 5 lattice points (a_i, b_i) $i = 1, 2, 3, 4, 5$, the pigeonhole principle implies that at least three of the integers $a_1, \ldots, a_5$ have the same parity. Without loss of generality, assume a_1, a_2, and a_3 have the same parity. Now by the pigeonhole principle, at least two of the points b_1, b_2, b_3 must have the same parity. Without loss of generality, assume b_1 and b_2 have the same parity. Now the midpoint of the segment from (a_1, b_1) to (a_2, b_2) is $(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2})$, and this is a lattice point since a_1 and a_2 have the same parity and b_1 and b_2 have the same parity.

12. In the worst case, each of the 12 charging stations would have a 7 vehicles before the next vehicle would give one charging stations 8 vehicles. Thus, $12 \times 7 + 1 = 85$ vehicles charging simultaneously would guarantee that one station has 8 vehicles.

14. (a) Partition the balls into "50-sum" sets $\{1, 49\}, \{2, 48\}, \ldots, \{24, 26\}$ and two unpaired singleton $\{25\}$ and $\{50\}$. This gives 26 sets. If balls are

drawn and assigned to the appropriate set, to insure that one set receives two balls, we must draw 27 balls.

Chapter 3

Number Theory

3.1. Divisibility

6. If $d \mid n^2$, then it need not be true that $d \mid n$. For example, $4|6^2$ but $4 \not| 6$.

9. (a) Suppose $a|b$. Then $b = na$ for some $n \in \mathbb{Z}$. If $c \in D_a$, then $c|a$, so $a = mc$ for some $m \in \mathbb{Z}$, so $b = na = n(mc) = (nm)c$ where $nm \in \mathbb{Z}$, so $c|b$, that is, $c \in D_b$. Thus, $D_a \subseteq D_b$. Conversely, suppose $D_a \subseteq D_b$. Now $a \in D_a$ so $a \in D_b$, and thus $a|b$.

   (b) Suppose $a|b$. Then $b = na$ for some $n \in \mathbb{Z}$. If $c \in M_b$, then $c = mb$ for some $m \in \mathbb{Z}$, so $c = mb = m(na) = (mn)a$ where $mn \in \mathbb{Z}$, so $a|c$. Thus, $c \in M_a$. This shows that $M_b \subseteq M_a$. Conversely, suppose $M_b \subseteq M_a$. Since $b \in M_b$, we have $b \in M_a$, so $a|b$.

   (c)
$$\begin{aligned} D_a = D_b \quad &\iff \quad D_a \subseteq D_b \text{ and } D_b \subseteq D_a \\ &\iff \quad a|b \text{ and } b|a \quad \text{by part (a)} \\ &\iff \quad a = \pm b \quad \text{(Theorem 3.1.7)} \\ &\iff \quad |a| = |b| \end{aligned}$$

   (d)
$$\begin{aligned} M_a = M_b \quad &\iff \quad M_a \subseteq M_b \text{ and } M_b \subseteq M_a \\ &\iff \quad b|a \text{ and } a|b \quad \text{by part (b)} \\ &\iff \quad a = \pm b \quad \text{(Theorem 3.1.7)} \\ &\iff \quad |a| = |b| \end{aligned}$$

12. (a) $a = 73$, $b = 25$: $q = 0, r = 25$.
    (b) $a = 25$, $b = 73$: $q = 2, r = 23$.
    (c) $a = -73$, $b = -25$: $q = 1, r = 48$.
    (d) $a = -25$, $b = -73$: $q = 3, r = 2$.
    (e) $a = 79$, $b = -17$: $q = -1, r = 62$.
    (f) $a = -17$, $b = 79$: $q = -4, r = 11$.

(g) $a = -37$, $b = 13$: $q = 0, r = 13$.
(h) $a = 13$, $b = -37$: $q = -3, r = 2$.

21. (a) If a and b leave a remainder of 2 when divided by 7, then $a = 7q + 2$ and $b = 7s + 2$ for some integers q and s, and thus $a - b = 7q + 2 - (7s + 2) = 7(q - s)$ where $q - s \in \mathbb{Z}$, so $7|(a - b)$.
(b) If $a = 7q + 2$, then $10a = 70q + 20 = 70q + 14 + 6 = 7(10q + 2) + 6$. Thus, by uniqueness of the quotient and remainder when $10a$ is divided by 7, we have a quotient of $10q + 2$ and a remainder of 6.
(c) If $a = 7q + 2$, then $100a = 700q + 7(28) + 4 = 7(100q + 28) + 4$, so by the uniqueness of remainders, $100a$ leaves a remainder of 4 when divided by 7.

25. We will show $4|(13^n - 1)\ \forall n \in \mathbb{N}$ by mathematical induction. If $n = 1$, then $4|(13^1 - 1)$ since $4|12$. Suppose $4|(13^k - 1)$. We wish to show that $4|(13^{k+1} - 1)$. Now

$$\begin{aligned} 13^{k+1} - 1 &= 13(13^k - 1 + 1) - 1 \\ &= 13(13^k - 1) + 13 - 1 \\ &= 13(13^k - 1) + 12 \end{aligned}$$

Now $4|(13^k - 1)$ by the induction hypothesis and $4|12$, so $4|(13^{k+1} - 1)$. Now by mathematical induction, $4|(13^n - 1)\ \forall n \in \mathbb{N}$.

Alternatively, the result of Exercise 28 shows that $12|(13^n - 1)\ \forall n \in \mathbb{N}$, and since $4|12$, we have $4|(13^n - 1)\ \forall n \in \mathbb{N}$.

3.2. The Euclidean Algorithm

1. (a) $\gcd(561, 330) = 33 = 3(561) - 5(330)$.
(b) $\gcd(3542, 276) = 46 = 13(276) - 3542$.
(c) $\gcd(2145, 663) = 39 = 13(663) - 4(2145)$.

6. If $a, b \in \mathbb{Z}$ and z and w are linear combinations of a and b using integer coefficients, say $z = ja + kb$ and $w = la + mb$ $(j, k, l, m \in \mathbb{Z})$ then a linear combination of z and w with integer coefficients has form $sz + tw$ $(s, t \in \mathbb{Z})$. Now

$$\begin{aligned} sz + tw &= s(ja + kb) + t(la + mb) \\ &= (sj + tl)a + (ks + tm)b \end{aligned}$$

is a linear combination of a and b with integer coefficients $sj + tl$ and $ks + tm$.

8. $\gcd(15, 39) = 3$, so 3 should divide $15s + 39t$ for any $s, t \in \mathbb{Z}$. The bill should be of form $15s + 39t$ $(s, t \in \mathbb{N} \cup \{0\})$, so the bill should be a multiple of 3 cents. It is not.

13. Suppose $a, b, q, r \in \mathbb{Z} \setminus \{0\}$ and $a = bq + r$.
(a) $\gcd(a, b) = \gcd(b, r)$ is true.
Proof: If d is any common divisor of a and b, then d is a divisor of $a - bq = r$. Thus, any common divisor of a and b is a common divisor of b and r. Conversely, any common divisor of b and r must also divide b and $bq + r = a$, and therefore must be a common divisor of a and b.

This shows that the common divisors of a and b are precisely the common divisors of b and r, so $\gcd(a,b) = \gcd(b,r)$.

(c) In general, $\gcd(q,r)$ does not divide b. For example, with $a = 45, b = 7, q = 6$, and $r = 3$, we have $\gcd(q,r) = 3$ but 3 does not divide 7.

3.3. The Fundamental Theorem of Arithmetic

3. Substituting the expressions for $\text{lcm}(m,n)$ and $\gcd(m,n)$ given in Corollary 3.3.4 into $mn = \gcd(m,n)\text{lcm}(m,n)$ and observing that $\min\{m_i,n_i\}+\max\{m_i,n_i\} = m_i + n_i$ proves Corollary 3.3.5.

11. (a) If $n = p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}$, then $n^k = p_1^{kn_1}p_2^{kn_2}\cdots p_j^{kn_j}$, and it follows that $m = n^k$ is a perfect k^{th} power if and only if the multiplicity of each prime factor of m is a multiple of k.

(b) Suppose $m = p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}$. If m is a perfect square, then $2|n_i$ for each $i = 1,\ldots,j$. If m is a perfect cube, then $3|n_i$ for each $i = 1,\ldots,j$. If m is simultaneously a perfect square and a perfect cube, then $2|n_i$ and $3|n_i$ for each i, so the prime factorization of each n_i contains a 2 and a 3. Thus, $6|n_i$ for each $i = 1,\ldots,j$, and it follows that m is a perfect 6^{th} power.

14.

$$\begin{aligned} d|a \quad &\Rightarrow \quad a = dq \text{ for some } q \in \mathbb{Z} \\ &\Rightarrow \quad a^2 = d^2q^2 \text{ for } q^2 \in \mathbb{Z} \\ &\Rightarrow \quad d^2|a^2. \end{aligned}$$

Conversely, suppose $d^2|a^2$. Then $a^2 = d^2s$ for some $s \in \mathbb{Z}$. Consider the prime factorization of $s = \frac{a^2}{d^2}$. If the prime factorizations of a^2 and d^2 are $p_1^{2n_1}\cdots p_j^{2n_j}$ and $p_1^{2m_1}\cdots p_j^{2m_j}$ respectively, then by dividing we find that the prime factorization of $s = \frac{a^2}{d^2}$ must be $p_1^{2(n_1-m_1)}\cdots p_j^{2(n_j-m_j)} = t^2$ where $t = p_1^{n_1-m_1}\cdots p_j^{n_j-m_j} = \frac{a}{d}$. Now since $s = t^2$ is a perfect square, $a^2 = d^2s \Rightarrow a^2 = d^2t^2 \Rightarrow a = \pm dt \Rightarrow d|a$.

31. Suppose $p(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0$ is a polynomial with integer coefficients $c_0,\ldots,c_n$, and $r = \frac{a}{b}$ $(a,b \in \mathbb{Z}, b \neq 0, \gcd(a,b) = 1)$ is a rational number with $p(r) = 0$. Since $p(r) = 0$, we have

$$\frac{c_na^n}{b^n} + \frac{c_{n-1}a^{n-1}}{b^{n-1}} + \cdots + \frac{c_2a^2}{b^2} + \frac{c_1a}{b} + c_0 = 0. \tag{3.1}$$

Multiplying both sides of this equation by b^n and rearranging the terms gives

$$c_na^n = -c_{n-1}a^{n-1}b - \cdots - c_2a^2b^{n-2} - c_1ab^{n-1} - c_0b^n.$$

Since b divides the right hand side of this equation, it must divide the left hand side, so $b|c_na^n$. Since $\gcd(a,b) = 1$, we have $\gcd(b,a^n) = 1$ and thus $b|c_n$.

Again multiplying Equation (3.1) by b^n and rearranging the terms, we find that

$$c_na^n + c_{n-1}a^{n-1}b + \cdots + c_2a^2b^{n-2} + c_1ab^{n-1} = -c_0b^n.$$

Since a divides the left hand side of this equation, a must divide the right hand side as well, so $a|c_0b^n$. Since $\gcd(a,b) = 1 = \gcd(a,b^n)$, it follows that $a|c_0$.

Together with the result of the previous paragraph, this proves the Rational Root Theorem.

32. (a) To count the number of factors of form $(a_i - a_j)$ where $1 \le i < j \le m+1$, observe that once i is selected, the inequality $i < j \le m+1$ implies that there are $m+1-i$ possibilities for j. As i may assume any value from 1 to m, the number of factors is $\sum_{i=1}^{m}(m+1-i) = m+(m-1)+\cdots+2+1 = T_m$.
    (b) The pigeonhole principle implies that $\lceil \frac{m+1}{2} \rceil$ of the integers $a_1, \ldots a_{m+1}$ must have the same parity. The argument of (a) shows that from any s integers $b_1, \ldots, b_s$, we may form T_{s-1} distinct factors $(b_i - b_j)$. Thus, the $\lceil \frac{m+1}{2} \rceil$ integers from $a_1, \ldots, a_{m+1}$ of the same parity give $k = T_{\lceil \frac{m+1}{2} \rceil - 1}$ distinct even factors $(a_i - a_j)$ in P, and it follows that $2^k | P$.

3.4. Divisibility Tests

2. (a) $10a = 110q + 10r = 110q + 11r - r = 11(10q + r) - r = 11q' - r$ where $q' = 10q + r \in \mathbb{Z}$.
   (b) The case $m = 0$ is clear: $10^0 a = a = 11q + (-1)^0 r$ as given. If $10^k a = 11q' + (-1)^k r$, then applying (a) gives $10(10^k a) = 11q'' - (-1)^k r$ or $10^{k+1} a = 11q'' + (-1)^{k+1} r$ for some $q'' \in \mathbb{Z}$. By mathematical induction, $10^m a = 11q'' + (-1)^m r$ for any integer $m \ge 0$.
   (c) In $a = 11q + r$, take $a = 1, q = 0$, and $r = 1$, so that for any integer $m \ge 0$, $10^m = 11q'' + (-1)^m$ for some integer q''.

5. If $s = \langle d_{2j-1} \cdots d_2 d_1 d_0 \rangle$ has $2j$ digits then $t = \langle d_0 d_1 d_2 \cdots d_{2j-1} \rangle$, so

$$\begin{aligned}
s + t &= (d_{2j-1}10^{2j-1} + d_{2j-2}10^{2j-2} + \cdots + d_2 10^2 + d_1 10 + d_0) \\
&\quad +(d_0 10^{2j-1} + d_1 10^{2j-2} + \cdots + d_{2j-3}10^2 + d_{2j-2}10^1 + d_{2j-1}) \\
&= d_{2j-1}(10^{2j-1} + 1) + d_{2j-2}(10^{2j-2} + 10^1) \\
&\quad +d_{2j-3}(10^{2j-3} + 10^2) + \cdots + d_j(10^j + 10^{j-1}) \\
&\quad +d_{j-1}(10^{j-1} + 10^j) + \cdots + d_2(10^2 + 10^{2j-3}) \\
&\quad +d_1(10^1 + 10^{2j-2}) + d_0(1 + 10^{2j-1}) \\
&= d_{2j-1}(10^{2j-1} + 1) + 10d_{2j-2}(10^{2j-3} + 1) \\
&\quad +10^2 d_{2j-3}(10^{2j-5} + 1) + \cdots + 10^{j-1}d_j(10^1 + 1) \\
&\quad +10^{j-1}d_{j-1}(1 + 10^1) + \cdots + 10^2 d_2(1 + 10^{2j-5}) \\
&\quad +10d_1(1 + 10^{2j-3}) + d_0(1 + 10^{2j-1}).
\end{aligned}$$

Recalling that $11|(10^m + 1)$ for any odd number m, and observing that each term in the last expression above contains a factor of form $(10^m + 1)$ (m odd), we have $11|(s + t)$.

11. For $n = \langle d_k d_{k-1} \cdots d_2 d_1 d_0 \rangle$, we have

$$\begin{aligned}
n &= \langle d_k d_{k-1} \cdots d_3 000 \rangle + 100d_2 + 10d_1 + d_0 \\
&= \langle d_k d_{k-1} \cdots d_3 000 \rangle + 96d_2 + 4d_2 + 8d_1 + 2d_1 + d_0 \\
&= [\langle d_k d_{k-1} \cdots d_3 000 \rangle + 96d_2 + 8d_1] + [4d_2 + 2d_1 + d_0].
\end{aligned}$$

Since 8 divides each term in the first bracketed expression above, 8 divides that bracketed expression, and it follows from Corollary 3.4.4 that $8|n$ if and only if 8 divides the second bracketed expression. That is, $8|n$ if and only if $8|(4d_2 + 2d_1 + d_0)$.

14. All of the tests below are direct consequences of Theorem 3.4.5.
    (a) For $n = \langle d_k \cdots d_2 d_1 d_0 \rangle$, $12|n$ if and only if $[3|(d_k + \cdots + d_1 + d_0)$ and $4|\langle d_1 d_0 \rangle]$. Proof: $12|n$ if and only if $3|n$ and $4|n$, since 3 and 4 are relatively prime.
    (b) For $n = \langle d_k \cdots d_2 d_1 d_0 \rangle$, $14|n$ if and only if $[2|d_0$ and $7|n]$. Proof: $14|n$ if and only if $2|n$ and $7|n$, since 2 and 7 are relatively prime.
    (c) For $n = \langle d_k \cdots d_2 d_1 d_0 \rangle$, $15|n$ if and only if $[5|d_0$ and $3|(d_k + \cdots + d_1 + d_0)]$. Proof: $15|n$ if and only if $5|n$ and $3|n$, since 5 and 3 are relatively prime.
    (d) For $n = \langle d_k \cdots d_2 d_1 d_0 \rangle$, $18|n$ if and only if $[9|(d_k + \cdots + d_1 + d_0)$ and $2|d_0]$.
    Proof: $18|n$ if and only if $9|n$ and $2|n$, since 9 and 2 are relatively prime.
    (e) For $n = \langle d_k \cdots d_2 d_1 d_0 \rangle$, $75|n$ if and only if $[25|\langle d_1 d_0 \rangle$ and $3|(d_k + \cdots + d_1 + d_0)]$. Proof: $75|n$ if and only if $25|n$ and $3|n$, since 25 and 3 are relatively prime.

23. Suppose that the sum of the digits of a and the sum of the digits of $5a$ both equal k. Then $a = 9q + k$ and $5a = 9n + k$ for some integers q, n. Now $4a = 5a - a = 9(n - q)$, so $9|4a$. Since 9 and 4 are relatively prime, we have $9|a$.

3.5. Number Patterns

2.

$$\begin{aligned}
\frac{1 + 3 + \cdots + (2n-1)}{(2n+1) + \cdots + (4n-1)} &= \frac{1 + 3 + \cdots + (2n-1)}{(1 + 3 + \cdots + (4n-1)) - (1 + 3 + \cdots + (2n-1))} \\
&= \frac{n^2}{(2n)^2 - n^2} \\
&= \frac{n^2}{3n^2} \\
&= \frac{1}{3}
\end{aligned}$$

4. (a) We will use mathematical induction. For $n = 1$, the statement $1 \cdot 2 = \frac{1(1+1)(1+2)}{3}$ is clearly true. Suppose that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

We wish to show that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}.$$

Now
$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)(k+2)$

$$\begin{aligned}
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \text{ by the induction hypothesis} \\
&= (k+1)(k+2)\left(\frac{k}{3}+1\right) \\
&= (k+1)(k+2)\left(\frac{k+3}{3}\right),
\end{aligned}$$

as needed. Now by mathematical induction, $1\cdot2+2\cdot3+3\cdot4+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$ $\forall n \in \mathbb{N}$.

(b) $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots+n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ $\forall n \in \mathbb{N}$. For $n = 1$, the statement is clearly true. Suppose that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}.$$

Then
$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$$\begin{aligned}
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= (k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right) \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}.
\end{aligned}$$

Now by mathematical induction, the result holds for all $n \in \mathbb{N}$.

(c) $1 \cdot 2 \cdots j + 2 \cdot 3 \cdots (j+1) + \cdots (n)(n+1) \cdots (n+j-1) = \frac{n(n+1)\cdots(n+j)}{j+1}$ for all $j, n \in \mathbb{N}$. Suppose $j \in \mathbb{N}$ is given. The case $n = 1$ is clearly true. If $1\cdot2\cdots j+2\cdot3\cdots(j+1)+\cdots(k)(k+1)\cdots(k+j-1) = \frac{k(k+1)\cdots(k+j)}{j+1}$, then
$1\cdot2\cdots j+2\cdot3\cdots(j+1)+\cdots(k)(k+1)\cdots(k+j-1)+(k+1)(k+2)\cdots(k+j)$

$$\begin{aligned}
&= \frac{k(k+1)\cdots(k+j)}{j+1} + (k+1)(k+2)\cdots(k+j) \\
&= (k+1)\cdots(k+j)\left(\frac{k}{j+1}+1\right) \\
&= (k+1)\cdots(k+j)\left(\frac{k+j+1}{j+1}\right),
\end{aligned}$$

as needed. Now by mathematical induction, the result holds for all $n \in \mathbb{N}$. Since $j \in \mathbb{N}$ was arbitrary, this completes the proof.

6. By adding 1 to each odd number in the n^{th} row of Nicomachus' Pattern, we obtain the n^{th} row of this pattern. Since there are n terms in the n^{th} row of Nicomachus' Pattern, we find that the sum of the n^{th} row of this pattern exceeds the corresponding sum in Nicomachus' Pattern by n, and is thus n^3+n.

   Alternatively, observe that the sum of the first n rows of this pattern is the sum of the first T_n even numbers, namely $2+4+\cdots+2T_n = 2(1+2+\cdots T_n) = 2T_{T_n}$. Now the sum of the entries in the n^{th} row alone is the sum of the first

n rows minus the sum of the first $n-1$ rows. That is, the sum of the entries in the n^{th} row is

$$\begin{aligned} 2(T_{T_n} - T_{T_{n-1}}) &= T_n(T_n+1) - T_{n-1}(T_{n-1}+1) \\ &= T_n^2 + T_n - T_{n-1}^2 - T_{n-1} \\ &= (T_n - T_{n-1})(T_n + T_{n-1}) + (T_n - T_{n-1}) \\ &= n(n^2) + n \\ &= n^3 + n. \end{aligned}$$

8.
$$\begin{aligned} 1 &= 1 \\ 3+5 &= 8 \\ 6+9+12 &= 27 \\ 10+14+18+22 &= 64 \\ &\vdots \end{aligned}$$

$$\begin{aligned} T_n + (T_n+n) + (T_n+2n) + \cdots + (T_n+(n-1)n) &= nT_n + [n + 2n + \cdots + (n-1)n] \\ &= nT_n + nT_{n-1} \\ &= n(T_n + T_{n-1}) \\ &= n(n^2) \\ &= n^3 \end{aligned}$$

11. (a) $(4, 12, 24, 40, \ldots) = (4T_1, 4T_2, 4T_3, 4T_4, \ldots)$.
(b) $(4T_n - n)^2 + \cdots + (4T_n - 1)^2 + (4T_n)^2 = (4T_n + 1)^2 + \cdots + (4T_n + n)^2$, or $\sum_{j=0}^{n}(4T_n - j)^2 = \sum_{j=1}^{n}(4T_n + j)^2$.
(c)
$$\begin{aligned} \sum_{j=0}^{n}(4T_n - j)^2 &= \sum_{j=0}^{n}(16T_n^2 - 8jT_n + j^2) \\ &= \sum_{j=0}^{n}(16T_n^2 + j^2) - 8T_n\sum_{j=0}^{n} j \\ &= (16T_n^2 + 0^2) + \sum_{j=1}^{n}(16T_n^2 + j^2) - 8T_n^2 \\ &= \sum_{j=1}^{n}(16T_n^2 + j^2) + 8T_n^2 \\ &= \sum_{j=1}^{n}(16T_n^2 + j^2) + 8T_n\sum_{j=1}^{n} j \\ &= \sum_{j=1}^{n}(16T_n^2 + 8jT_n + j^2) \\ &= \sum_{j=1}^{n}(4T_n + j)^2. \end{aligned}$$

13.

$$\begin{aligned}(1, 3, \ldots, (2n-1)) \cdot (n, n, \cdots, n) &= 1n + 3n + \cdots + (2n-1)n \\ &= n(1 + 3 + \cdots + (2n-1)) \\ &= n(n^2) \\ &= n^3\end{aligned}$$

Chapter 4

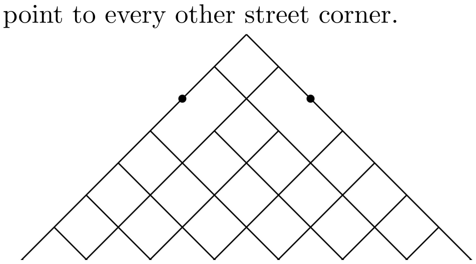
Combinatorics

4.1. Getting from Point A to Point B

2. $\binom{7}{4} = 35$.

4. Since $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, a divisor of 2310 having exactly 3 prime factors will be of form $p_1p_2p_3$ where $p_1, p_2, p_3 \in \{2, 3, 5, 7, 11\}$. There are $\binom{5}{3} = 10$ ways to pick three primes p_1, p_2, p_3 from the set $\{2, 3, 5, 7, 11\}$, and thus there are 10 divisors of 2310 having exactly three prime factors.

7. For each map shown below, find the number of southerly paths from the top point to every other street corner.



(a)

1							
1	1						
1	2	1					
1	2	2	1				
1	3	4	3	1			
1	4	7	7	4	1		
1	5	11	14	11	5	1	
1	6	16	25	25	16	6	1

11. For $n = 1, \ldots, 6$, the table below shows the routes from A to B which make exactly n turns. Each route is seven blocks, three of which are to the west (denoted by w) and four of which are to the east (denoted by e). The numbers in the bottom row of the table show the number of routes from A to B which

make exactly n turns.

1	2	3	4	5	6
wwweeee	wweeeew	wweweee	weweeew	wewewee	ewewewe
eeeewww	ewwweee	wweewee	ewwewee	weweewe	
	weeeeww	wweeewe	ewweewe	weewewe	
	eewwwee	wewweee	weeweew	eweweew	
	eeewwwe	ewweeew	ewewwee	eweewew	
		weewwee	eweewwe	eewewew	
		weeewwe	weeewew		
		eweeeww	eewwewe		
		eewweew	eewewwe		
		eeweeww			
		eeeweww			
		eeewwew			
2	5	12	9	6	1

4.2. The Fundamental Principle of Counting

1. A positive divisor of $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$ has form $p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_k^{m_k}$ where $0 \le m_j \le n_j$ for each $j = 1, \ldots, k$. Thus, the number of positive divisors of n is the number of ways to choose a sequence $(m_1, \ldots, m_k)$ of whole numbers satisfying $0 \le m_j \le n_j$ for each $j = 1, \ldots, k$. As there are $n_j + 1$ choices for m_j $(j = 1, \ldots, k)$, the Fundamental Principle of Counting tells us that there are $(n_1+1)(n_2+1)\cdots(n_k+1)$ positive divisors of n. Applying this to $2^3 3^2 7^1 11^1$, we see that there are $(4)(3)(2)(2) = 48$ positive divisors of $2^3 3^2 7^1 11^1$.

3. There are two choices—depressed or not—for each of the four valves, so there are $2^4 = 16$ fingering positions for a four-valve instrument. Equivalently, each fingering position corresponds to a subset of valves to be depressed. There are 4 valves and $2^4 = 16$ possible subsets.

   If the third and fourth valves are not to be depressed simultaneously, then there are three choices for positions of the third and fourth valves: only the third valve depressed, only the fourth valve depressed, or neither valve depressed. These 3 options follow the 2 options (depressed or not) for the first valve and the 2 options (depressed or not) for the second valve. This gives a total of $2 \cdot 2 \cdot 3 = 12$ fingering positions in which the third and fourth valve are not depressed simultaneously.

6. There are $\binom{7}{4}$ ways to select the Democrats and $\binom{9}{4}$ ways to select the Republicans, so there are $\binom{7}{4}\binom{9}{4} = 35 \cdot 126 = 4410$ ways to make the appointments.

7. For her birthday, Camila receives 4 wrapped gifts from her family and 7 wrapped gifts from friends.

   (f) In how many orders may she unwrap five gifts, exactly two of which are from her family?

   The tasks are: select a set of 2 gifts from 4, select 3 gifts from 7, then order the 5 gifts. This can be done in $\binom{4}{2}\binom{7}{3}5! = (6)(35)(120) = 25,200$ ways.

   (g) In how many orders may she unwrap 2 gifts from her family first and then 3 gifts from friends?

The tasks are: choose an ordered list of 2 from 4, choose an ordered list of 3 from 7. There are $P(4,2)P(7,3) = (4)(3)(7)(6)(5) = 2,520$ outcomes.

11. (a) There are 26 choices for the first letter, 26 choices for the second letter, 26 choices for the third letter, 10 choices for the first digit, 10 choices for the second digit, and 10 choices for the third digit, for a total of $26^3 10^3 = 17,576,000$ possible license plates.
    (b) $26^3 10^3 \binom{6}{3} = 351,520,000$; There are 26 choices for each of the three letters, 10 choices for each of the three digits, and $\binom{6}{3}$ ways to choose three of the six positions for the letters.

4.3. A Formula for the Binomial Coefficients

4. (c) $\frac{2^n \cdot (2^n-1)(2n-2)}{6} = \binom{2^n}{3}$
   (e) $2 \cdot 239 \cdot 238 \cdot 237 \cdot 236 = \frac{240 \cdot 239 \cdot 238 \cdot 237 \cdot 236}{5!} = \binom{240}{5}$
7. Ashley has a pack of 96 crayons, all of different colors.
   (a) In how many ways may Ashley write her name using a different color for each letter? $P(96,6) = (96)(95)(94)(93)(92)(91) = 667,474,778,880$.
   (b) In how many ways may Ashley write her name using a single color for each letter, but possibly with repeated colors? $96^6 = 782,757,789,696$
   (c) In how many ways may Ashley pick a set of 6 crayons of different colors to take to her desk? $\binom{96}{6} = 927,048,304$
9. (a) $P(11,3) = 11 \cdot 10 \cdot 9 = 990$
   (b) $P(11,3)P(11,3)P(10,3)P(8,3) = (11 \cdot 10 \cdot 9)(11 \cdot 10 \cdot 9)(10 \cdot 9 \cdot 8)(8 \cdot 7 \cdot 6) = 237,105,792,000$
14. (a) $\binom{52}{5} = 2,598,960$
    (b) $\binom{4}{3}\binom{4}{2} = 24$

4.4. Permutations with Indistinguishable Objects

1. The frequency of letters in each anagram are given, followed by an application of Theorem 4.4.2.
   (a) c, 1; o,1; m, 2; i,1; t,2; e,2; s,1; $\frac{10!}{2!2!2!} = 453,600$
   (b) m,2; e,3; a,1; s, 2; u,1; r,1; n,1; t,1; $\frac{12!}{2!2!3!} = 19,958,400$
   (c) t,1; h,1; e,3; p,1; r,3; o,2; f,1, a,1; d,1, s,1; $\frac{15!}{2!3!3!} = 18,162,144,000$
   (d) r,1; e,3; v,1; i, 3; s,1; d,2; t,1; o,1; n,1; $\frac{14!}{2!3!3!} = 1,210,809,600$
   (e) t,2; h,1; e, 4; o,4; d,1; r,2; s,1, v,1; l,1; $\frac{17!}{2!2!4!4!} = 154,378,224,000$
   (f) t,3; r,2; u,1; s,3; w,1; o,1; h,1; i,1; n,1; e,1; $\frac{15!}{2!3!3!} = 18,162,144,000$
   (g) w,1; i,2; l,2; a,3; m,1; s,2; h,1; k,1; e,3; p,1; r,1; $\frac{18!}{2!2!2!3!3!} = 22,230,464,256,000$
   (h) t,4; h,2; e,6; u,3; n,1; i,3; d,1; s,4; a,2; b,1; r,2; o,1; f,2; $\frac{32!}{2!2!2!2!3!3!4!4!6!} = 1,101,524,811,141,375,548,928,000,000$
3. There are $\frac{15!}{6!6!3!} = 420,420$ distinguishable permutations of the six indistinguishable nut crunch bars, six indistinguishable chocolate bars, and three indistinguishable toffee bars, and thus there are 420,420 distinguishable ways to distribute the bars to a row of 15 students.

6. Let F represent a football toss ticket, C a cakewalk ticket, and G a miniature golf ticket. The number of indistinguishable arrangements of the tickets

F F F C C C C G G G G G G

is $\frac{13!}{3!4!6!} = 60,060$.

11. Find the multinomial coefficients indicated.
    (c) The coefficient of $x^{12}y^4z^6w^4$ in the expansion of $(x+y+z+w)^{26}$ is $\binom{26}{12,4,6,4} = \frac{26!}{12!4!6!4!} = 2030145117000$.
    (d) The coefficient of $x^5y^2z^7$ in the expansion of $(2x+3y+z)^{14}$.
    With $a = 2x, b = 3y, c = z$, the coefficient of $a^5b^2z^7$ in the expansion of $(a+b+c)^{14}$ is $\binom{14}{5,2,7} = \frac{14!}{5!2!7!} = 72072$, as in part (a). Thus, in the multinomial expansion $(2x+3y+z)^{14}$, we have the term $\binom{14}{5,2,7}(2x)^5(3y)^2z^7 = \binom{14}{5,2,7}2^53^2x^5y^2z^7$, so the coefficient of $x^5y^2z^7$ is $\binom{14}{5,2,7}2^53^2 = 20756736$.

13. How many distinguishable arrangements of the letters of BOOKKEEPER have
    (a) a string of five vowels in a row?
    There are $\frac{5!}{3!}2!$ ways to form a string of five adjacent vowels O, O, E, E, E, and then if V represents this string, there are $\frac{6!}{2!}$ ways to arrange V, B, K, K, P, R. Completing both tasks can be done in $\frac{5!}{3!}2!\frac{6!}{2!} = 3600$ ways.
    (b) no adjacent Es?
    The letters B, O, O, K, K, P, R can be arranged in $\frac{7!}{2!2!}$ ways. From the 8 positions (before or after any of these seven letters) where we could place an E, we must choose 3, in $\binom{8}{3}$ ways. All together, there are $\frac{7!}{2!2!}\binom{8}{3} = 70,560$ outcomes.

4.5. Combinations with Indistinguishable Objects

2. (a) Each child has 7 choices. $7^5 = 16,807$.
    (b) Five drinks can be placed in 11 drink-or-divider slots in $\binom{11}{5} = 462$ ways.

8. (a) The budget increase will be divided into 100 equal one-percent increments which will be distributed among three areas. This may be done in $\binom{100+3-1}{100} = \binom{102}{2} = 5151$ ways.
    (b) After 15% increases are distributed to each of the three areas, there remain 55 one-percent increments to be divided among the three areas. This can be done in $\binom{55+3-1}{2} = \binom{57}{2} = 1596$ ways.
    (c) After a 50% increase is allotted for salaries, the remaining 50 one-percent increments can be distributed to the three areas in $\binom{50+3-1}{2} = \binom{52}{2} = 1326$ ways.

10. An urn contains 6 balls marked with "A", 7 balls marked with "B", and 4 balls marked with "C".
    (a) The 17 balls are distributed to 17 people so that each person gets exactly one ball. How many outcomes are possible?
    The number of distinguishable arrangements of AAAAAA BBBBBBB CCCC is $\frac{17!}{6!7!4!} = 4,084,080$.
    (b) The 6 balls marked "A" are distributed to 17 people, with no restriction on how many any person gets. How many outcomes are possible?

Six stars (As) interspersed among 16 dividers can be accomplished in $\binom{22}{6} = 74,613$ ways.

(c) Carol randomly chooses four balls from the 17. How many outcomes are possible, if the order of her selection is not relevant?
This is like selecting 4 donuts of three types, A, B, and C. Four stars (balls) and 2 bars can be arranged in $\binom{6}{2} = 15$ ways.

(d) How many balls must be drawn from the 17 to guarantee: (i) two balls with the same letter? (ii) two balls with "B"? (iii) four balls with the same letter?
(i) By the pigeonhole principle, 4. (ii) In the worst case, all 6 As and 4 Cs will be drawn before the second B, requiring $6+4+2 = 12$ balls. (iii) By the strong form of the pigeonhole principle, $3 \cdot 3 + 1 = 10$ balls will be required. The worst case would be to draw AAABBBCCC and not yet have four of the same letter, but the next ball would give four of some letter.

14. Beck has six identical boxes of crayons, each containing 48 different colors.
    (a) How many ways could Beck choose 6 crayons from his supply?
    This problem is like choosing 6 donuts from an adequate supply (at least 6 of each) of 48 types. This corresponds to 6 stars and 47 bars, giving $\binom{53}{6} = 22,957,480$ outcomes.
    (b) How many ways could Beck choose up to 6 crayons from his supply?
    We partition the problem into the selections with 0, 1, 2, 3, 4, 5, and 6 crayons and work each of these as in (a). In each case, there will be 47 dividers, but now 0, 1, 2, 3, 4 ,5, or 6 stars. This leads to

$$\sum_{k=0}^{6} \binom{47+k}{k} = \binom{47}{0} + \binom{48}{1} + \binom{49}{2} + \binom{50}{3} + \binom{51}{4} + \binom{52}{5} + \binom{53}{6}.$$

This comes to $25,827,165$.

4.6. The Inclusion-Exclusion Principle

7. How many permutations of the letters e, x, c, l, u, s, i, o, n include "cel", "in", or "on"?

   Let S_{cel} be the set of permutations of the letters containing "cel", with S_{in} and S_{on} defined analogously. Now S_{cel} contains all permutations of the seven items cel, x, u, s, i, o, n, so $|Scel| = 7!$. S_{in} contains all permutations of the eight items *in, e, x, c, l, u, s, o*, so $|S_{in}| = 8!$. Similarly, $|S_{on}| = 8!$. $S_{cel} \cap S_{in}$ consists of all permutations of the six items cel, in , x, u, s, o, so $|S_{cel} \cap S_{in}| = 6!$. Similarly, $|S_{cel} \cap S_{on}| = 6!$. $S_{in} \cap S_{on} = \emptyset$ since there is only one n and it can have at most one predecessor, and thus $|S_{in} \cap S_{on}| = |S_{cel} \cap S_{in} \cap S_{on}| = 0$. Thus, $|S_{cel} \cup S_{in} \cup S_{on}| = 7! + 8! + 8! - (6! + 6! + 0) - 0 = 84,240$.

10. As a part of a two-factor authentication system, a company sends the user a four-digit code (a_1, a_2, a_3, a_4). How many such codes have no two consecutive digits a_i, a_{i+1} with $a_{i+1} = 1 + a_i$? (Thus, $(4, 9, 0, 3)$ is permitted, but $(3, 7, 8, 3)$ is not.)

    It is easier to count which of the $10^4 = 10000$ codes contain a consecutive pair of digits with $a_{i+1} = 1 + a_i$. For $k = 1, 2, 3$, let $A_k = \{(a_1, a_2, a_3, a_4) \in$

$\{0,\ldots,9\}^4 : a_{k+1} = 1+a_k\}$. The codes in A_k have 9 choices (namely, $0,\ldots,8$) for a_k; a_{k+1} is uniquely determined by the choice of a_k; and there are 10 choices each for the other two digits of the code. Thus, $|A_k| = 9(10)^2 = 900$. $A_1 \cap A_2$ contains all the codes $(a_1, 1+a_1, 2+a_1, a_4)$. There are 8 choices $(0,\ldots,7)$ for a_1 and 10 choices for a_4, so $|A_1 \cap A_2| = 80$. Similarly, $|A_2 \cap A_3| = 80$. The codes in $A_1 \cap A_3$ have form $(a_1, 1+a_1, a_3, 1+a_3)$, and there are 9 choices $(0,\ldots,8)$ each for a_1 and a_3, so $|A_1 \cap A_3| = 9^2 = 81$. Finally, $A_1 \cap A_2 \cap A_3$ contains the codes $(a_1, 1+a_1, 2+a_1, 3+a_1)$, and there are 7 choices $(0,\ldots,6)$ for a_1, so $|A_1 \cap A_2 \cap A_3| = 7$. Thus, the number we want is $10^4 - |A_1 \cup A_2 \cup A_3| = 10000 - (900+900+900) + (80+80+81) - 7 = 7534$.

14. An autumn mix of jelly beans contains orange, red, yellow, and brown beans. If five beans are chosen at random and put into a bag, how many outcomes will have at least one of each color in the bag?

    $\sim$(at least one of each color) = no orange OR no red OR no yellow OR no brown. Let O, R, Y, and B be the set of outcomes with no orange, no red, no yellow, and no brown, respectively. Since each of the five beans could be any of the four colors, there are $4^5 = 1024$ ways to put five beans in the bag. Of these 1024, $|O \cup R \cup Y \cup B|$ of them do not contain at least one of each color. Now $|O| = |R| = |Y| = |B| = 3^5$, since these selections of beans require five beans from any of the three "other" colors. If A and B are any two of the sets, $|A \cap B| = 2^5$ since these are the ways to choose 5 beans from the two "other" colors, and similarly $|A \cap B \cap C| = 1^5$ if A, B, C are any of the three sets. Furthermore, $|O \cap R \cap Y \cap B| = 0$. Thus, $|O \cup R \cup Y \cup B| = \binom{4}{1}3^5 - \binom{4}{2}2^5 + \binom{4}{3}1 + \binom{4}{4}0 = 784$ is the number which do not contain at least one color. Thus, $1024 - 784 = 240$ outcomes contain at least one color.

19. Twelve people put their name in a hat to possibly participate in a gift exchange. Each person then draws a name out of the hat, and if they draw a name other than their own, they give that person a gift. In how many ways may the twelve names be drawn out of the hat? In how many ways are exactly nine gifts exchanged?

    There are $12! = 479{,}001{,}600$ assignments of the 12 names to 12 people. If exactly nine gifts are exchanged, then exactly three people drew their own name. There are $\binom{12}{3} = 220$ ways to select which three drew their own name, and then the remaining nine names must be deranged. The number of derangements of nine items is $9!(1 - 1 + 1/2! - 1/3! + 1/4! - \cdots - 1/9!) = 133496$. Thus, there are $220 * 133496 = 29{,}369{,}120$ ways exactly nine gifts are exchanged.

4.7. Circular Permutations

4. From twelve distinct beads, nine are to be strung in a loop for a necklace, and the remaining beads are to be suspended in a straight line from one of the beads on the loop. Use circular permutations to count the number of possible outcomes. Your answer should suggest another solution which does not use circular permutations. Give such an argument.

   From 12 beads, choose the 9 to be in the loop, then form an undirected circular permutation of those nine, then select one of the beads in the loop to suspend the remaining three from, and finally determine the linear order in which these three beads will be suspended. These steps can be completed in $\binom{12}{9} \cdot \frac{8!}{2} \cdot 9 \cdot 3! = \frac{12!}{9!3!} \frac{9!3!}{2} = \frac{12!}{2}$ ways.

12! is the number of ways to arrange the 12 beads in a linear order. From any linear order of the beads on a string, looping the string after the 12th bead back and tying it to the gap between the 3rd and 4th bead gives a necklace of 9 beads with 3 suspended from one bead. However, the permutation abcd(efghijkl) gives the same necklace as abcd(lkjihgfe), so we divide by two to compensate for the undirected nature of the necklace. Similarly, from a necklace as described, starting at the bottom suspended bead, tracing up to the loop, and around the loop (either clockwise or counterclockwise) gives a map from necklaces to linear orders..

8. A Ferris wheel has 12 cars, each of which holds up to two people seated side by side.
   (a) Even though the Ferris wheel may turn in both directions, explain why this problem involves directed circular permutations.
   The occupants are seated in a specified direction. They will be facing the backs of the same people in front of them regardless of whether they are moving forward or backward. Similarly, the cars have a front and back, so even for unoccupied cars, they are directed.
   (b) Twelve individuals are seated, one in each car. How many ways may they be seated?
   This is a directed circular permutation of 12 objects. 11!.
   (c) Twelve couples are seated, one couple in each car. How many ways may they be seated?
   The only distinction in this problem and (b) is that once the couples are assigned to a car, there are 2 ways to determine who sits on the left. Thus, there are $11!2^{12} = 163,499,212,800$ outcomes.
   (d) 24 people are seated. How many ways are possible?
   From the 24 people, choose the 12 who will sit on the left, then for each of those 12, choose who will sit to their right. This can be done in $\binom{24}{12}12!$ ways. Then arrange the 12 pairs in a directed circular permutation in 11! ways. There are $\binom{24}{12}12!11! = 51,704,033,477,769,953,280,000$ outcomes.

9. Twelve people sit around a round table. How many (undirected) seating arrangements are possible if
   (a) A cannot sit by B or C.
   Create a seating block $\_A\_$. There are $9 \cdot 8 = 72$ ways to select A's right and left neighbor. Now string the remaining 9 individual beads and one superbead $\_A\_$ in an undirected circular permutation in $9!/2$ ways, giving a total of $9 \cdot 8 \cdot 9!/2 = 13,063,680$ possible outcomes.
   (b) None of A, B, or C can be adjacent.
   Create seating blocks $A\_$, $B\_$, and $C\_$ in $9 \cdot 8 \cdot 7$ ways. String the three blocks and six remaining individuals around the table like 9 beads, in $8!/2$ ways. This gives $9(8)(7)8!/210,160,640$ possible outcomes.
   (c) A, B, and C must sit together.

There are 3! ways to linearly arrange A, B, and C, but since ABC and some permutation of the other people will be counted in circular permutations in both directions, we do not need to count CBA as a different arrangement of A, B, C. Thus, there are $3!/2 = 3$ possible undirected arrangements of A, B, C. (There are three choices for which of A, B, or C sits in the middle.) Then string this $\{A, B, C\}$ block and the 9 remaining individuals in an undirected circular permutation in $9!/2$ ways, giving $3 * 9!/2 = 544,320$ outcomes.

10. From 16 distinct beads, ten will be strung in a circular loop and linear strands of three beads will be suspended below two of the beads on the loop.
    (a) How many necklaces are possible if the linear strands are suspended from adjacent beads?
    Tracing the beads from the bottom of one linear strand up to the loop, around the loop, and down the other linear strand gives a linear permutation. There are 16! such linear permutations, but each permutation and its reverse permutation give the same necklace, so there are $16!/2$ such necklaces.
    (b) How many necklaces are possible?
    Method 1: Choose the three beads for the first linear strand hanging below the loop and order them, choose the 3 beads for the second strand and order them, string the remaining 10 beads in a undirected circular loop, choose the beads from which to suspend the first and second strands.

$$\binom{16}{3}3!\binom{13}{3}3!\left(\frac{9!}{2}\right)10 \cdot 9 = 94,152,554,496,000.$$

Method 2: Create "quadruple beads" to constitute the suspended strands including the bead on the loop, then string the single and quadruple beads. Choose the 4 beads for the first strand and order them, choose the 4 bead for the second strand and order them, string the 10 beads in undirected circular permutations.

$$\binom{16}{4}4!\binom{12}{4}4!\left(\frac{9!}{2}\right) = 94,152,554,496,000.$$

Method 3: This method is based on method used for (a). Consider the number of beads on the shortest path between the suspended strands. Counting the beads on the loop, each strand accounts for 4 beads, so there remain 8 beads not in a strand. If the strands are adjacent as in (a), there are 0 beads between the strands on the short path from strand to strand and 8 beads on the long path from strand to strand. Focusing on the short path between the strands, there could be 0, 1, 2, 3, or 4 beads between the strands. Suppose there are k beads in the shortest path between strands. Tracing the beads from the bottom of one linear strand up to the loop, around the long path between the strands, down the other strand, and then adding the k beads in order from the base of the second strand to the base of the first strand gives a linear permutation of the 16 beads. Reversing the permutation does not give an equivalent necklace, so there are 16! necklaces for each value of $k = 0, 1, 2,$ or 3. However, for $k = 4$, the shortest path between the strands has the same length as the longest path. Every permutation in which $ABCD$ is counted as the shortest path and $EFGH$ is counted as the longest path is indistinguishable if $ABCD$ and $EFGH$ are interchanged, so 16! double counts these, and there are only 16!/2 necklaces with four beads between the strands. Adding for $k = 0, 1, 2, 3, 4$, we get $4(16!) + 16!/2 = 16!(9/2) = 94,152,554,496,000$.

4.8. Probability

2. (a) $\frac{1}{435}$. There are $\binom{30}{2} = 435$ possible pairs, and {Sarah, Becky} constitute only one such pair.
   (b) $\frac{28}{435}$. Of the $\binom{30}{2} = 435$ possible pairs, there are 28 of form {Sarah, x} where x is a member other than Sarah or Becky.
   (c) $\frac{57}{435}$. Of the $\binom{30}{2} = 435$ possible pairs, there are 28 in which Sarah is selected but not Becky (see (b)) and likewise, 28 in which Becky is selected but not Sarah. Together with one pair in which both are selected, this gives $28 + 28 + 1 = 57$ pairs including Sarah or Becky.
   (d) $\frac{378}{435}$. From (c), 57 of the 435 pairs include Sarah or Becky, so the remaining $435 - 57 = 378$ pairs include neither Sarah nor Becky.

4. Note that the sample space S consists of $\binom{52}{5} = 2,598,960$ possible 5-card hands.
   (a) $\frac{4\binom{13}{5}}{\binom{52}{5}} = \frac{5148}{2{,}598{,}960} \approx 0.00198079$. There are 4 possible suits, and once the suit is selected, $\binom{13}{5}$ possible hands within that suit.
   (b) $\frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{624}{2{,}598{,}960} \approx 0.000240096$. From 13 kinds, we choose 1. From the 4 of this kind, we choose all 4, and from the 48 cards not of this kind, we choose 1.
   (c) $\frac{13 \cdot 12\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2{,}598{,}960} \approx 0.00144058$. There are 13 ways to choose the kind to get 3 of, and $\binom{4}{3}$ ways to select the three from 4 cards of this kind, and There are 12 ways to choose the kind to get 2 of, and $\binom{4}{2}$ ways to select the two from 4 cards of this kind.
   (d) $\frac{\binom{10}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} == \frac{10240}{2{,}598{,}960} \approx 0.00394004$. There are 10 choices (A, 2, 3, ..., 10) for the lowest card in the straight. This determines which 5 values will be in the straight. There are 4 cards of each of these values and we wish to choose 1 of each.

   We have found that there are
   624 ways to get a four of a kind
   3744 ways to get a full house
   5148 ways to get a flush,and
   10240 ways to get a straight.
   Thus, these hands are here listed in order from rarest to most common, so four of a kind beats a full house, a full house beat a flush, and a flush beats a straight.

8. $\frac{30}{1200} = \frac{1}{40}$. Since $1200 = 2^4 3^1 5^2$, any positive divisor of 1200 has form $2^r 3^s 5^t$ where $0 \le r \le 4$, $0 \le s \le 1$, and $0 \le t \le 2$. As there are 5 choices for r, 2 choices for s, and 3 choices for t, there are $(5)(2)(3) = 30$ divisors of 1200 in the set $\{1, 2, \ldots, 1200\}$.

10. As seen in Example 4.8.4, the sample space contains $\binom{19}{7} = 50,388$ elements.
    (b) $\frac{19,305}{50,388}$. There are $\binom{13}{5}$ ways to select the five colors. Since there must be one gumball of each color, this accounts for 5 gumballs. The remaining 2 may be distributed among the 5 colors (requiring 4 dividers) in $\binom{2+4}{2} = \binom{6}{2} = 15$ ways. Thus, there are $1287 \cdot 15 = 19,305$ assortments with exactly 5 colors.
    (c) $\frac{19,071}{50,388}$. No more than 4 colors means 1 color, 2 colors, 3 colors, or 4 colors. In (a) we found that there are 13 assortments with one color, and in Example 4.8.4 we found that there are 14300 assortments with exactly 4 colors.

    Assortments with exactly 2 colors: There are $\binom{13}{2} = 78$ ways to choose the 2 colors. Since there must be one gumball of each color, this accounts for 2 gumballs. The remaining 5 may be distributed among the 2 colors (requiring 1 divider) in $\binom{5+1}{5} = \binom{6}{5} = 6$ ways. Thus, there are $78 \cdot 6 = 468$ assortments with exactly 2 colors.

    Assortments with exactly 3 colors: There are $\binom{13}{3} = 286$ ways to choose the 3 colors. Since there must be one gumball of each color, this accounts for 3 gumballs. The remaining 4 may be distributed among the 3 colors (requiring 2 dividers) in $\binom{4+2}{4} = \binom{6}{4} = 15$ ways. Thus, there are $286 \cdot 15 = 4290$ assortments with exactly 3 colors.

    Combining our results, there are $13 + 468 + 4290 + 14300 = 19,071$ assortments with no more than 4 colors.

Relations

5.1. Relations

3. If $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (1, 3)\}$, then R is not symmetric (for $(1, 3) \in R$ but $(3, 1) \notin R$) and is not antisymmetric (for $(1, 2) \in R$ and $(2, 1) \in R$, but $1 \neq 2$). This shows that neither implication holds.

7. The ordered pairs given with some negative answers suggest points at which the property in question fails.

Relation	Domain	Range	Refl.	Sym.	Antisym.	Trans.
(a) S	{1,3,5}	{3,5}	No	No	No	No: (3,5), (5,3)
(b) R	$\mathbb{N}$	$\mathbb{N} \setminus \{1\}$	No (1,1)	No (1,3)	No (7,8)	No (100,15),(15,5)
(c) T	{0,4,7}	{0,4,7}	Yes	Yes	No (0,7)	Yes
(d) U	$\mathbb{Z} \setminus \{0\}$	$\mathbb{Z} \setminus \{0\}$	No (0,0)	Yes	No (1,2)	Yes
(e) P	$\mathbb{Z} \setminus \{0\}$	$\mathbb{Z} \setminus \{0\}$	No (5,5)	Yes	No (3,7)	No (6,5), (5,2)

10. (a) $S \times S$ has 9 elements, and thus has $2^9 = 512$ subsets. Relations on S are subsets of $S \times S$, so there are 512 relations on S.
    (b) Every reflexive relation on S has form $\{(1, 1), (2, 2), (3, 3)\} \cup C$ where C is a subset of the remaining six elements of $S \times S$. There are 2^6 such subsets C, and thus $2^6 = 64$ reflexive relations on S.

(c) The relations described are of form

$$\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\} \cup C$$

where C is a subset of the remaining three elements of $S \times S$, that is, where $C \subseteq \{(2,1),(3,1),(3,2)\}$. There are $2^3 = 8$ such subsets. They are $C_1 = \emptyset$, $C_2 = \{(2,1)\}$, $C_3 = \{(3,1)\}$, $C_4 = \{(3,2)\}$, $C_5 = \{(2,1),(3,1),\}$, $C_6 = \{(2,1),(3,2)\}$, $C_7 = \{(3,1),(3,2)\}$, and $C_8 = \{(2,1),(3,1),(3,2)\}$. Now the 8 such relations are given by $R_i = \{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\} \cup C_i$ for $i = 1,\ldots,8$.

(d) Each relation R_i $(i = 1,\ldots 8)$ has $\{1,2,3\}$ as domain and range.

(e)

	Refl.	Sym.	Antisym.	Trans.
R_1	Yes	No	Yes	Yes
R_2	Yes	No	No	Yes
R_3	Yes	No	No	No
R_4	Yes	No	No	Yes
R_5	Yes	No	No	No
R_6	Yes	No	No	No
R_7	Yes	No	No	No
R_8	Yes	Yes	No	Yes

14. (a) (i) {(2,3), (2,1), (3,5), (4,4)}
(ii) {(1,3), (3,5), (5,4), (5,2)}
(iii) {(5,4), (5,2)}

(b) The graph of S is a parabola in $\mathbb{R}^2$ with vertex at the origin and having the y-axis as axis of symmetry. The graph of $S|_{[0,\infty)}$ is the right half of that parabola, including the vertex $(0,0)$.

(c) (i) {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6) }
(ii) {(2,2), (2,3), (2,4), (2,5), (2,6), (4,4), (4,5), (4,6)}
(iii) {(6,6)}

17. (a) $R_1 \circ R_n = R_1$ and $R_n \circ R_1 = R_1$ for all $n \in \{1,2,\ldots,16\}$.

(b) $R_8 \circ R_n = R_n$ and $R_n \circ R_8 = R_n$ for all $n \in \{1,2,\ldots,16\}$.

(c) For all $n \in \{1,2,\ldots,16\}$, $R_{16} \circ R_n$ is the largest relation on $\{1,2\}$ having the same domain as R_n, and $R_n \circ R_{16}$ is the largest relation on $\{1,2\}$ having the same range as R_n.

5.2. Equivalence Relations

5. Define a relation on $\mathbb{Z}$ by $a \sim b$ if and only if $5|(a^2 - b^2 + 5)$. Show that $\sim$ is an equivalence relation, and find all equivalence classes.

For any $a \in \mathbb{Z}, a \sim a$ since $5|(0+5)$. $a \sim b \Rightarrow 5|(a^2 - b^2 + 5) \Rightarrow 5|(-a^2 + b^2 - 5) \Rightarrow 5|(b^2 - a^2 - 5 + 10) \Rightarrow b \sim a$, so $\sim$ is symmetric. $a \sim b, b \sim c \Rightarrow 5|(a^2 - b^2 + 5) \wedge 5|(b^2 - c^2 + 5) \Rightarrow 5|(a^2 - b^2 + 5 + b^2 - c^2 + 5) \Rightarrow 5|(a^2 - c^2 + 10) \Rightarrow 5|(a^2 - c^2 + 5) \Rightarrow a \sim c$. Thus, $\sim$ is transitive. Note that $5|(a^2 - b^2 + 5) \iff 5|(a^2 - b^2)$. Thus,

$$[0] = \{n \in \mathbb{Z} : 0 \sim 5\} = \{n \in \mathbb{Z} : 5|(0^2 - n^2)\} = \{n \in \mathbb{Z} : 5|n\} = \{5n : n \in \mathbb{Z}\}.$$

$[1] = \{n \in \mathbb{Z} : 1 \sim 5\} = \{n \in \mathbb{Z} : 5|(1^2 - n^2)\} = \{n \in \mathbb{Z} : 5|(1-n)(1+n)\} = \{n \in \mathbb{Z} : 5|(n-1) \vee 5|(n+1)\} = \{n \in \mathbb{Z} : (n \pm 1 = 5k(\text{some} k \in \mathbb{Z}\} = \{5\text{k} \pm 1 : \text{k} \in \mathbb{Z}\}$.

$[2] = \{n \in \mathbb{Z} : 2 \sim 5\} = \{n \in \mathbb{Z} : 5|(2^2 - n^2)\} = \{n \in \mathbb{Z} : 5|(n-2)(n+2)\} = \{n \in \mathbb{Z} : 5|(n-2) \vee 5|(n+2)\} = \{n \in \mathbb{Z} : (n \pm 2 = 5k(\text{some} k \in \mathbb{Z}\} = \{5\text{k} \pm 2 : \text{k} \in \mathbb{Z}\}$.

Now $\{[0], [1], [2]\}$ partition $\mathbb{Z}$, so these are all the equivalence classes.

9. $[\frac{\pi}{3}] = \{\frac{\pi}{3} + 2\pi n : n \in \mathbb{Z}\}$. $[\frac{\pi}{2}] = \{\frac{\pi}{2} + 2\pi n : n \in \mathbb{Z}\}$. $[\pi] = \{(2n+1)\pi : n \in \mathbb{Z}\}$.

10. The smallest equivalence relation R on $\{1,2,3,4,5\}$ with $\{(1,1),(1,2),(2,4),(3,1)\} \subseteq R$ is $R = (\{1,2,3,4\} \times \{1,2,3,4\}) \cup \{(5,5)\}$. $3R1$ and symmetry imply $1R3$. $1R2$ and $2R4$ imply $1R4$. Thus, $1Rx$ for $x \in \{1,2,3,4\}$. $1R2$ and symmetry imply $2R1$, and then since $1Rx$ for all $x \in \{1,2,3,4\}$, transitivity gives $2Rx$ for $x \in \{1,2,3,4\}$. Similarly, $3R1$ implies $3Rx$ for $x \in \{1,2,3,4\}$. Now $1R2, 2R4 \Rightarrow 1R4 \Rightarrow 4R1 \rightarrow 4Rx$ for $x \in \{1,2,3,4\}$. Finally, $5R5$ by reflexivity.

14. (a) $\sim$ is not reflexive and not transitive, and thus is not an equivalence relation.
    (b) $\approxeq$ is an equivalence relation. The equivalence classes are $\{\emptyset\}$, $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2\},\{1,3\},\{2,3\}\}$, and $\{\{1,2,3\}\}$.
    (c) $\simeq$ is an equivalence relation. The equivalence classes are $\{\emptyset,\{2\}\}$, $\{\{1\},\{1,2\}\}$, $\{\{3\},\{2,3\}\}$, and $\{\{1,3\},\{1,2,3\}\}$.
    (d) $\approx$ is not symmetric and is thus not an equivalence relation.

5.3. Partial Orders

3. (a) The relation is a partial order.
   (b) $\sqsubseteq$ is reflexive since $x \leq x^2\ \ \forall x \in \mathbb{N}$. $\sqsubseteq$ is not antisymmetric. For example, $7 \sqsubseteq 8$ and $8 \sqsubseteq 7$, yet $7 \neq 8$. $\sqsubseteq$ is not transitive. For example, $15 \sqsubseteq 4$ and $4 \sqsubseteq 2$, but $15 \not\sqsubseteq 2$. Thus $\sqsubseteq$ is not a partial order.
   (c) The relation $\ll$ is reflexive, but is neither antisymmetric ($3 \ll 4$ and $4 \ll 3$ but $3 \neq 4$, for example) nor transitive ($6 \ll 4$ and $4 \ll 2$, but $6 \not\ll 2$, for example). Thus, $\ll$ is not a partial order.

5. (a) a is maximum and thus maximal. d and g are minimal. There is no minimum element.
   (b) $lub\{c,a\} = a$. In general, $\text{lub}\{x,y\} = y$ if and only if $y \geq x$.
   (c) $\text{glb}\{d,x\}$ does not exist for $x \in \{c,e,f,g\}$.

9. $(\mathcal{P}(S), \subseteq)$ is totally ordered if and only if $|S| \leq 1$.

   If $|S| = 0$, then $\mathcal{P}(S) = \{\emptyset\}$, a one-element collection totally ordered by inclusion. If $|S| = 1$, then $\mathcal{P}(S) = \{\emptyset, S\}$, a two-element collection totally ordered by inclusion. If $|S| \geq 2$, then there exist distinct elements $a, b \in S$, and $\{a\},\{b\} \in \mathcal{P}(S)$ but $\{a\} \nsubseteq \{b\}$ and $\{b\} \nsubseteq \{a\}$. Thus, if $|S| \geq 2$, then $(\mathcal{P}(S), \subseteq)$ is not totally ordered.

17. (a) Yes. The maximum element of S is an upper bound of C.
    (b) No. Let $P = [0,1) \cup (2,3]$ in $\mathbb{R}$ with the usual order. The upper bounds of $C = [0,1)$ are precisely the points of $(2,3]$. Thus, $C = [0,1)$ has upper bounds, but no least upper bound.

(c) No. Let $P = \{\{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$ ordered by inclusion, and let $C = \{\{a\}, \{b\}\}$. Now the set of upper bounds of C is $UB = \{\{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$. Now since UB has no minimum element, C has no least upper bound.

(d) Yes. If C has a least upper bound, then the set UB of upper bounds of C is nonempty, and if UB has a minimum element, it must be unique. (See Theorem 5.3.6.)

19. $a \prec b$ if and only if there is a line from a upward to b in the Hasse diagram for the poset.

(a) $\{2\} \prec \{2, 3\}$; $\{3\} \prec \{2, 3\}$; $\{2, 3\} \prec \{2, 3, 4\}$; $\{4, 5\} \prec \{4, 5, 6\}$.

(b) $(-1, -1) \prec (0, 0)$; $(1, -1) \prec (0, 0)$; $(0, 0) \prec (-1, 1)$; $(0, 0) \prec (1, 1)$; $(-1, 1) \prec (-2, 2)$; $(-1, 1) \prec (0, 2)$; $(-1, 1) \prec (2, 2)$; $(1, 1) \prec (-2, 2)$; $(1, 1) \prec (0, 2)$; $(1, 1) \prec (2, 2)$

(c) $(-2, 2) \prec (-1, 1)$; $(-2, 2) \prec (-1, -1)$; $(-1, 1) \prec (0, 2)$; $(-1, 1) \prec (0, 0)$; $(-1, -1) \prec (0, 0)$; $(-1, -1) \prec (0, 2)$; $(0, 0) \prec (1, 1)$; $(0, 0) \prec (1, -1)$; $(0, 2) \prec (1, 1)$; $(0, 2) \prec (1, -1)$; $(1, 1) \prec (2, 2)$; $(1, -1) \prec (2, 2)$

(d) $2 \prec 5$; $3 \prec 5$; $5 \prec 6$; $4 \prec 6$

20. $\prec$ is never reflexive since, from the definition of the covering relation, a cannot cover itself.

$\prec$ is sometimes symmetric, but only in a poset (such as $(\mathbb{R}, \leq)$) in which $\{(a, b) | a \prec b\} = \emptyset$.

$\prec$ is sometimes transitive. If $a \prec b$ and $b \prec c$, then $a \not\prec c$; however, the defining implication for transitivity

$$[a \prec b \text{ and } b \prec c] \Rightarrow a \prec c \qquad \text{for all } a, b, c$$

will be true if the antecedent is false. That is, $\prec$ will be transitive if there are no elements a, b, c with $a \prec b$ and $b \prec c$.

$\prec$ is always (vacuously) antisymmetric, for $[a \prec b$ and $b \prec a]$ is false for any choice of a and b, so the implication $[a \prec b$ and $b \prec a] \Rightarrow a = b$ is a true implication.

$\|$ is never reflexive, for $a \leq a \ \forall a \in P$. (We assume $P \neq \emptyset$.)

$\|$ is always symmetric, for if $a \not\leq b$ and $b \not\leq a$, then $b \not\leq a$ and $a \not\leq b$.

$\|$ is sometimes transitive. The following are equivalent: (i) $\|$ is transitive, (ii) $(P, \leq)$ has no noncomparable elements, and (iii) $(P, \leq)$ is a totally ordered set. We will show $\sim$(ii) $\Rightarrow \sim$(i): If $(P, \leq)$ has noncomparable elements a and b, then $a \parallel b$ and $b \parallel a$, yet $a \not\parallel a$, so $\|$ is not transitive. Next, we will show $\sim$(i) $\Rightarrow \sim$(ii): If $\|$ is not transitive, then there must exist $a, b, c \in P$ with $a \parallel b$, $b \parallel c$, yet $a \not\parallel c$; in particular, there exist noncomparable elements $a, b \in P$ (or $b, c \in P$). We have now shown (i) $\iff$ (ii). The equivalence of (ii) and (iii) is immediate.

$\|$ is sometimes antisymmetric. If $(P, \leq)$ has a pair of noncomparable elements a, b (i.e., if P is not totally ordered), then $a \neq b$ yet $a \parallel b$ and $b \parallel a$, so $\|$ is not antisymmetric. However, if $(P, \leq)$ is totally ordered, the implication $[a \parallel b$ and $b \parallel a] \Rightarrow a = b$ is vacuously true (since the antecedent is never true). Thus, $\|$ is antisymmetric if and only if $(P, \leq)$ is totally ordered.

25. (a) Yes. If each P_i has a maximum element m_i, then $(m_i)_{i\in I}$ is the maximum element in P, for given any $(x_i)_{i\in I} \in P$, we have $x_i \leq_i m_i \ \forall i \in I$, so by the definition of the product order, $(x_i)_{i\in I} \leq (m_i)_{i\in I}$.

(b) Yes. Suppose $(m_i)_{i\in I}$ is the maximum element in P. Then $\forall (x_i)_{i\in I} \in P$, we have $(x_i)_{i\in I} \leq (m_i)_{i\in I}$ and hence $x_i \leq_i m_i \ \forall i \in I$. We claim m_{i_0} is the maximum element of P_{i_0}. Suppose $x \in P_{i_0}$. Define $(a_i)_{i\in I} \in P$ by $a_i = m_i$ for $i \in I \setminus \{i_0\}$ and $a_{i_0} = x$. Now $(a_i)_{i\in I} \leq (m_i)_{i\in I}$ implies $x = a_{i_0} \leq m_{i_0}$. Since $x \in P_{i_0}$ was arbitrary, this shows that m_{i_0} is maximum in P_{i_0}.

(c) If, for all $i \in I$, m_i is a maximal element in P_i, then $(m_i)_{i\in I}$ is maximal in P, for if $(x_i)_{i\in I} \in P$ with $(x_i)_{i\in I} \geq (m_i)_{i\in I}$, then $m_i \leq_i x_i \ \forall i \in I$. Since m_i is maximal in P_i, this implies $m_i = x_i \ \forall i \in I$, so $(x_i)_{i\in I} = (m_i)_{i\in I}$, and $(m_i)_{i\in I}$ is maximal in P.

If $(m_i)_{i\in I}$ is maximal in P, then m_i is maximal in $P_i \ \forall i \in I$, for if not, there exists $i_0 \in I$ and $x \in P_{i_0}$ with $m_{i_0} <_{i_0} x$. Now $(a_i)_{i\in I} \in P$ defined by $a_i = m_i \ \forall i \in I \setminus \{i_0\}$ and $a_{i_0} = x$ is a point of P strictly larger than $(m_i)_{i\in I}$, contrary to the maximality of $(m_i)_{i\in I}$.

5.4. Quotient Spaces

2. (a) For $0 < \epsilon < 2$, consider the line l_ϵ through the point $(2-\epsilon, 200) \in P(0,100)$ and the point $(2, 202-\epsilon) \in P(1,100)$. The slope of l_ϵ is $\frac{2-\epsilon}{\epsilon}$, and these slopes range from 0 to ∞ as ϵ ranges from 2 to 0. With arbitrary positive slopes allowed, we may find a line of this form passing through the point $(8, 2n+1) \in P(4,n)$ for every integer $n \geq 100$. Thus, $P(0,100)$, $P(1,100)$ and $P(4,n)$ are collinear for all integers $n \geq 100$. To show that $P(0,100)$, $P(1,100)$ and $P(4,n)$ are collinear for all integers $n < 100$, for $0 < \epsilon < 2$, consider the lines through $(2+\epsilon, 200) \in P(1,100)$ and $(2-\epsilon, 202-\epsilon) \in P(0,100)$. Such a line has slope $\frac{2-\epsilon}{-2\epsilon}$, and as ϵ ranges from 0 to 2, these slopes range from $-\infty$ to 0. Thus, there exists such a line through $P(0,100)$, $P(1,100)$ and $P(4,n)$ for all integers $n < 100$.

(c) No. Let $\mathcal{L}_1$ be the collinear set $\{P(0,100), P(2,102)\}$, let $\mathcal{L}_2$ be the collinear set $\{P(0,100), P(2,100)\}$, and let $\mathcal{L}_3$ be the collinear set $\{P(0,100), P(2,98)\}$. It is easy to see that (i) there is a line l_1 of slope m_1 which illuminates all the pixels of $\mathcal{L}_1$ if and only if $m_1 \in (\frac{1}{3}, 3)$, (ii) there is a line l_2 of slope m_2 which illuminates all the pixels of $\mathcal{L}_2$ if and only if $m_2 \in (-1, 1)$, and (iii) there is a line l_3 of slope m_3 which illuminates all the pixels of $\mathcal{L}_3$ if and only if $m_3 \in (-3, \frac{-1}{3})$.

Now $\mathcal{L}_1$ is parallel to $\mathcal{L}_2$ since there exist parallel lines l_i of slope $m_i = \frac{2}{3}$ illuminating all the pixels of $\mathcal{L}_i$ $(i = 1,2)$, and $\mathcal{L}_2$ is parallel to $\mathcal{L}_3$ since there exist parallel lines l_i of slope $m_i = \frac{-2}{3}$ illuminating all the pixels of $\mathcal{L}_i$ $(i = 2,3)$. However, no line l_1 illuminating all the pixels of $\mathcal{L}_1$ can be parallel to any line l_3 illuminating all the pixels of $\mathcal{L}_3$, since $m_1 \in (\frac{1}{3}, 3)$ and $m_3 \in (-3, \frac{-1}{3})$ imply $m_1 \neq m_3$.

7. Define an equivalence relation $\sim$ on $\mathbb{Z}$ by $a \sim b$ if and only if $a^2 \equiv b^2 \mod 3$. Find all the equivalence classes.

$[0] = \{b \in \mathbb{Z} : 3|b^2\} = \{b \in \mathbb{Z} : 3|b\} = \{3n : n \in \mathbb{Z}\}$.

Now it remains to find the equivalence class of integers of form $3n \pm 1$. Let us start with 1. $[1] = \{b \in \mathbb{Z} : 3|b^2 - 1\} = \{b \in \mathbb{Z} : 3|(b-1)(b+1)\} = \{b \in \mathbb{Z} : 3|(b-1) \text{ or } 3|(b+1)\}$. Now $3|(b-1) \iff b-1 = 3k \iff b = 3k+1$ for some $k \in \mathbb{Z}$ and $3|(b+1) \iff b+1 = 3k \iff b = 3k-1$ for some $k \in \mathbb{Z}$. Thus, $[1] = \{3k \pm 1 : k \in \mathbb{Z}\}$. Now $\mathbb{Z} = [0] \cup [1]$, so we have found all of the equivalence classes.

11. (a) Since $[3] \times [5] = [15] = [1]$ in $\mathbb{Z}/7$, $[5]$ is the multiplicative inverse of $[3]$ in $\mathbb{Z}/7$.
    (b) Since $[3] \times [2] = [6] = [1]$ in $\mathbb{Z}/5$, $[2]$ is the multiplicative inverse of $[3]$ in $\mathbb{Z}/5$.
    (c) Since $[3] \times [3] = [9] = [1]$ in $\mathbb{Z}/4$, $[3]$ is the multiplicative inverse of $[3]$ in $\mathbb{Z}/4$.
    (d) In $\mathbb{Z}/6$, We have $[3] \times [0] = [0]$, $[3] \times [1] = [3]$, $[3] \times [2] = [0]$, $[3] \times [3] = [3]$, $[3] \times [4] = [0]$, and $[3] \times [5] = [3]$. Thus, there is no $[n] \in \mathbb{Z}/6$ with $[3] \times [n] = [1]$, so $[3]$ has no multiplicative inverse in $\mathbb{Z}/6$.

15. (b) With $a \approx b$ if and only if $b - a \in \mathbb{N}$, $\oplus$ is well-defined. Suppose $[a] = [a']$ and $[b] = [b']$. Then $a = a' + j$ and $b = b' + k$ for some integers j, k. Now $a + b = a' + j + b' + k$, so $(a+b) - (a'+b') = j + k \in \mathbb{Z}$, and thus $a + b \approx a' + b'$. Now $[a] \oplus [b] = [a+b] = [a'+b'] = [a'] \oplus [b']$, so $\oplus$ is well-defined. $\otimes$ is not well-defined. $[1] = [2]$ and $[1.5] = [.5]$, but $[1] \otimes [1.5] = [(1)(1.5)] = [1.5] \neq [1] = [(2)(.5)] = [2] \otimes [.5]$.

19. On $\mathbb{R}^2$, define $(a, b) \lesssim (x, y)$ if and only if $b \leq y$. Describe the $\approx$-equivalence classes and the partial order on them induced by $\lesssim$.

    We have $(a, b) \approx (x, y)$ if and only if $b \leq y$ and $y \leq b$, which occurs if and only if $b = y$. Thus, $[(a, b)] = \{(x, b) : x \in \mathbb{R}\} = H_b$ is the horizontal line $y = b$. The partial order on $\{H_b : b \in \mathbb{R}\}$ is $H_a \leq H_b$ if and only if $a \leq b$.

21. (b) Reflexive: For any triangle t_1, t_1 has an angle whose measure is greater than or equal to that of every angle in t_1. Transitive: If t_1 has an angle whose measure is greater than or equal to that of every angle in t_2 and t_2 has an angle whose measure is greater than or equal to that of every angle in t_3, then t_1 has an angle whose measure is greater than or equal to that of every angle in t_3.

    Now $t_1 \lesssim t_2$ if and only if the largest angle in t_2 is greater than or equal to the largest angle in t_1. Thus, $t_1 \sim t_2$ if and only if the largest angle in t_1 has the same measure as the largest angle in t_2. The resulting partial order on equivalence classes is a total order.

27. (a) For any $a \in \emptyset$, we have $b \leq a \Rightarrow b \in \emptyset$ vacuously, so $\emptyset \in \mathcal{T}$. For $a, b \in S$, clearly $a \in S$ and $b \leq a$ implies $b \in S$, so $S \in \mathcal{T}$.
    (b) Suppose $D_1, D_2, \ldots, D_n \in \mathcal{T}$, $a \in D_1 \cap D_2 \cap \cdots \cap D_n$, and $b \leq a$. Since $D_1, D_2, \ldots, D_n$ are each decreasing, $b \in D_i (i = 1, 2, \ldots, n)$, so $b \in D_1 \cap \cdots \cap D_n$. Thus, $D_1 \cap \cdots \cap D_n \in \mathcal{T}$.
    (c) Suppose J is an arbitrary index set (finite or infinite), $D_j \in \mathcal{T}\ \forall j \in J$, $a \in \cup_{j \in J} D_j$, and $b \leq a$. Now $a \in \cup_{j \in J} D_j \Rightarrow \exists j_0 \in J$ such that $a \in D_{j_0}$. Now D_{j_0} decreasing implies $b \in D_{j_0}$, and thus $b \in \cup_{j \in J} D_j$. Thus $\cup_{j \in J} D_j \in \mathcal{T}$.

Chapter 6

Functions and Cardinality

6.1. Functions

2. (a) The graph represents a function from $\mathbb{R}$ to $\mathbb{R}$ if and only if every vertical line intersects the graph in exactly one point.
   If the graph represents a function f, the line $x = a$ intersects the graph in exactly one point $(a, f(a))$. If every vertical line $x = a$ intersects the graph in exactly one point, let this point be $(a, f(a))$ and this defines a function from $\mathbb{R}$ to $\mathbb{R}$.
   (b) The graph represents a function from a *subset* of $\mathbb{R}$ to $\mathbb{R}$ if and only if every vertical line intersects the graph in no more than one point.
   If the graph represents a function $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}$, then $x = a$ intersects the graph in exactly one point $(a, f(a))$ for each $a \in S$, and in no points for $a \notin S$. Conversely, if every vertical line intersects the graph G in no more than one point, the graph represents $f : S \to \mathbb{R}$ where $S = \{a \in \mathbb{R} | x = a \text{ intersects } G\}$ and $(a, f(a))$ is the intersection of $x = a$ with G.

6. Determine whether the functions below are one-to-one, onto, neither, or both. Jusitfy your answer.
   (b) $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = n^2 + n$ is one-to-one: $g(m) = g(n) \Rightarrow m^2 - n^2 + m - n = 0 \Rightarrow (m - n)(m + n + 1) = 0 \Rightarrow m = n$ since $m + n \neq -1$ for $m, n \in \mathbb{N}$. g is not onto since there is no natural number n with $g(n) = 5$, since the solutions to $n^2 + n - 5 = 0$ are irrational.
   (c) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ defined by $f(a, b) = 2a - b$ is not one-to-one: $f(10, 4) = 16 = f(9, 2)$. It is onto. For any nonnegative integer $k \geq 0$, $k = f(k + 1, k + 2)$ and $-k = f(1, k + 1)$. Since $(k + 1, k + 2), (1, k + 1) \in \mathbb{N} \times \mathbb{N}$, this shows every integer $k \geq 0$ or $-k \leq 0$ is in range, so f is onto.

12. We will show f is not one-to-one if and only if there is a horizontal line intersecting the graph more than once. If f is not one-to-one, then there exist $a \neq b$ such that $f(a) = f(b)$, so the horizontal line $y = f(a)$ intersects the

graph of f in (at least) two points, namely $(a, f(a))$ and $(b, f(b))$. Conversely, if a horizontal line $y = c$ intersects the graph of f in two points (a, c) and (b, c) with $a \neq b$, then $f(a) = f(b) = c$, so f is not one-to-one.

14. We will describe $f : \{1, 2, 3\} \to \{1, 2, 3\}$ by listing the ordered triple $(f(1), f(2), f(3))$. Now the functions $(1, 2, 3), (1, 3, 2), (2, 1, 3)$, and $(3, 2, 1)$ are the only ones with $f \circ f = id$. If $g \circ f$ is one-to-one, then f must be, so all functions with $f \circ f = id$ are one-to-one. (In fact, if $f \circ f = id$, then $f = f^{-1}$, so f is invertible and is thus one-to-one and onto.)

18. (a) $\sim_f$ is reflexive since $f(a) = f(a)\ \forall a \in A$. $\sim_f$ is symmetric since $f(a) = f(b) \Rightarrow f(b) = f(a)$. $\sim_f$ is transitive since $f(a) = f(b)$ and $f(b) = f(c)$ imply $f(a) = f(c)$.
    (b) If f is injective, then $a \sim_f b \iff f(a) = f(b) \iff a = b$, so $\sim_f$ is Δ_A. If $\sim_f$ is Δ_A and $f(a) = f(b)$, then $a \sim_f b$, so $(a, b) \in \Delta_A$, so $a = b$, and thus f is injective.

23. (a) Let $S = [-1, 0)$, $T = (0, 1]$, and $A = S \cup T$ have the usual order from $\mathbb{R}$. Consider $f : A \to \mathbb{R}$ (where $\mathbb{R}$ has the usual order) defined by $f(x) = x+1$ if $x < 0$ and $f(x) = x$ if $x > 0$. Now f is increasing on S and on T, but not on $A = S \cup T$ since, for example, $\frac{-1}{4} \leq \frac{1}{4}$ but $f(\frac{-1}{4}) = \frac{3}{4} \not\leq \frac{1}{4} = f(\frac{1}{4})$.
    (b) Suppose f is increasing on S and on T.
    $S \cap T \neq \emptyset$ is not necessary for f to be increasing on $A = S \cup T$: consider $f : S \cup T \to \mathbb{R}$ given by $f(x) = x$, where $S = [-1, 0)$ and $T = (0, 1]$ are subsets of $\mathbb{R}$ with the usual order.
    $S \cap T \neq \emptyset$ is not sufficient for f to be increasing on $A = S \cup T$: consider $A = S \cup T$ where $S = \{\{1\}, \{1, 2\}\}$ and $T = \{\{1\}, \{1, 2, 3\}\}$ with set inclusion as the order. Define $f : A \to \mathbb{N}$ (where $\mathbb{N}$ has the usual order) by $f(\{1\}) = 1$, $f(\{1, 2\}) = 5$, and $f(\{1, 2, 3\}) = 2$. Now f is increasing on S and on T but not on $A = S \cup T$ since $\{1, 2\} \subseteq \{1, 2, 3\}$ but $f(\{1, 2\}) = 5 \not\leq 2 = f(\{1, 2, 3\})$.

25. (a) Observe that $f \preceq g$ if and only if $\frac{f(1)-f(0)}{1-0} \leq \frac{g(1)-g(0)}{1-0}$, that is, if and only if the slope of f is less than or equal to the slope of g. Clearly $f \preceq f$ for any $f \in \mathcal{F}$ since the slope of f is less than or equal to the slope of f. If $f \preceq g$ and $g \preceq h$, then the slope of f is less than or equal to that of g, and the slope of g is less than or equal to that of h, so the slope of f is less than or equal to that of h, so $f \preceq h$, and thus $\preceq$ is transitive.

6.2. Inverse Relations and Inverse Functions

2. (a) The inverse relation {(2,1), (1,2), (4,3), (3,4)} is a function.
   (b) The inverse relation {(1,1), (1,3), (1,2), (1,4)} is not a function.
   (c) The inverse relation {(1,1), (2,1), (3,1), (4,1)} is a function.
   (d) The inverse relation {(3,1), (4,2), (3,3), (3,4)} is not a function.
   (e) The inverse relation {(3,1), (1,2), (2,3)} is not a function on {1,2,3,4}.

4. (a) $f^{-1}(x) = \frac{3x+1}{5}$
   (f) $f^{-1}(x) = \frac{1+2x}{x-2}$

5. (b) $T = [3, \infty)$, $f|_T^{-1}(x) = \sqrt{x+4}+3$, or $T' = (-\infty, 3]$, $f|_{T'}^{-1}(x) = -\sqrt{x+4}+3$

(d) $T = \mathcal{P}(\{2n-1|n \in \mathbb{N}\} \cup \{4n|n \in \mathbb{N}\})$ and $f|_T^{-1}(S) = \{s \in S|s \text{ is odd}\} \cup \{2s|s \in S \text{ and } s \text{ is even}\}$.

(e) $T = \mathbb{N} \cup \{0\}$ and $f|_T^{-1}(x) = x - 1$, or $T' = \{2n-1|n \in \mathbb{N}\} \cup \{2-2n|n \in \mathbb{N}\}$ and $f|_{T'}^{-1}(x) = x - 1$ if x is even and $f|_{T'}^{-1}(x) = 1 - x$ if x is odd.

(g) $T = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \ldots\}$ and $f|_T^{-1}(x) = \{1, 2, \ldots, x\}$ if $x \neq 0$; $f|_T^{-1}(0) = \emptyset$.

(h) $T = A$, $f(T) = \{[x,y] \subseteq \mathbb{R}|x \leq y+4\}$, and $f|_T^{-1}([x,y]) = [x+1, y-3]$.

16. Suppose $f : A \to B$ and $C \subseteq D \subseteq B$. Now

$$\begin{aligned} x \in f^{-1}(C) &\Rightarrow f(x) \in C \subseteq D \\ &\Rightarrow f(x) \in D \\ &\Rightarrow x \in f^{-1}(D). \end{aligned}$$

Thus, $f^{-1}(C) \subseteq f^{-1}(D)$.

The converse fails. If $f(x) = x^2$, then $f^{-1}([-10,1]) \subseteq f^{-1}([-3,1])$ but $[-10,1] \not\subseteq [-3,1]$.

24. (d) $f^{-1}(5)$ contains numbers of the following types:

TYPE	form/choices for each digit					total number
5 odd digits:						
	5 (odd)	5 (odd)	5 (odd)	5 (odd)	5 (odd)	$5^5 = 3125$
3 odd digits, 1 even:						
		4 (even $\neq 0$)	5 (odd)	5 (odd)	5 (odd)	$4 \cdot 5^3 = 500$
		5 (odd)	5 (even)	5 (odd)	5 (odd)	$5^4 = 625$
		5 (odd)	5 (odd)	5 (even)	5 (odd)	$5^4 = 625$
		5 (odd)	5 (odd)	5 (odd)	5 (even)	$5^4 = 625$
1 odd digit, 2 evens:						
			4 (even $\neq 0$)	5 (odd)	5 (even)	$4 \cdot 5^2 = 100$
			4 (even $\neq 0$)	5 (even)	5 (odd)	$4 \cdot 5^2 = 100$
			5 (odd)	5 (even)	5 (even)	$5^3 = 125$

Adding the numbers in the right column above gives $|f^{-1}(5)| = 5825$.

6.3. Cardinality of Infinite Sets

6. A function $f : \{1,2\} \to \mathbb{N}$ is completely characterized by the ordered pair $(f(1), f(2)) \in \mathbb{N} \times \mathbb{N}$. This gives a bijection between the set of all functions $f : \{1,2\} \to \mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$. Since $\mathbb{N}\times\mathbb{N}$ is countable, so is the set of all functions $f : \{1,2\} \to \mathbb{N}$.
7. (a) Suppose A_i is countable for $i = 1, 2, \dots, n$. Consider the products A_1, $A_1 \times A_2$, $A_1 \times A_2 \times A_3, \dots, \prod_{i=1}^{n} A_i$. Clearly A_1 is countable. If $\prod_{i=1}^{k} A_i$ is countable, then $\prod_{i=1}^{k+1} A_i = (\prod_{i=1}^{k} A_i) \times A_{k+1}$ is a product of two countable sets and is thus countable. By mathematical induction, it follows that $\prod_{i=1}^{n} A_i$ is countable for any n if each A_i is countable.
   (b) Corresponding $f : \{1, 2, \dots, n\} \to \mathbb{N}$ to $(f(1), f(2), \dots, f(n)) \in \mathbb{N}^n$ gives a bijection from the set B to $\mathbb{N}^n$. Since the latter set is countable by (a), the former set is also countable.
13. "The smallest natural number that cannot be defined using less than twenty words" is a thirteen-word description of that natural number.

6.4. An Order Relation on Cardinal Numbers

3. Each nondegenerate interval in $\mathbb{R}$ contains a rational number. Since there are only countably many rational numbers, there can be only countably many pairwise disjoint intervals.
4. Let $C_r' = C_r \cap \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$ be the first quadrant part of the circle C_r. The projection function $f : C_r' \to [0,r]$ defined by $f((x,y)) = x$ is a bijection. Since $[0,r]$ is uncountable (see Exercise 3 of Section 6.3), it follows that C_r' is uncountable. Now $C_r' = A \cup B \cup C$ where $A = \{(x,y) \in C_r' | x \in \mathbb{Q}\}$, $B = \{(x,y) \in C_r' | y \in \mathbb{Q}\}$, and $C = \{(x,y) \in C_r' | x \notin \mathbb{Q}, y \notin \mathbb{Q}\}$. Now A is indexed by a subset of $\mathbb{Q}$, so A is countable. ($h : A \to [0,r] \cap \mathbb{Q}$ defined by $h((x,y)) = x$ is a bijection.) Similarly, B is countable. Now $A \cup B \cup C = C_r'$ is uncountable, so C must be uncountable. Thus, $C_r' \subseteq C_r$ contains uncountably many points (x,y) with $x \notin \mathbb{Q}$ and $y \notin \mathbb{Q}$.
11. (a) No. $\frac{3}{2} = \frac{6}{4}$ but $g(\frac{3}{2}) = 2 \neq 4 = g(\frac{6}{4})$.
    (b) h is well defined since every element of $\mathbb{Q}^+$ has a unique representation as $\frac{m}{n}$ where m and n are relatively prime natural numbers. h is not one-to-one since $h(\frac{1}{4}) = 4 = h(\frac{3}{4})$. h is onto, since for any $n \in \mathbb{N}$, $n = h(\frac{1}{n})$.
16. Given an algebraic number α, pick a polynomial $p(x) = c_0 + c_1 x + \dots + c_n x^n$ with integer coefficients $c_0, \dots, c_n$ such that $p(\alpha) = 0$. (In fact, there exists a unique such polynomial of minimal degree such that $c_0, \dots, c_n$ are relatively prime and $c_n > 0$.) Suppose α is the m^{th} zero of $p(x)$ when the distinct real zeros of $p(x)$ are listed in increasing order. Map α to the natural number whose base 12 representation is the sequence of digits

$$m^1 m^2 \cdots m^j \pm c_0^1 c_0^2 \cdots c_0^{k_0} \pm c_1^1 c_1^2 \cdots c_1^{k_1} \cdots \pm c_{n-1}^1 c_{n-1}^2 \cdots c_{n-1}^{k_{n-1}} \pm c_n^1 c_n^2 \cdots c_n^{k_n}$$

where the digits base 12 are $0, 1, \dots, 9, +$, and $-$; $m = m^1 m^2 \cdots m^j$ where $m^1, \dots, m^j$ are the base 10 digits of m; and $c_i = \pm c_i^1 c_i^2 \cdots c_i^{k_i}$ where $c_i^1, \dots c_i^{k_i}$ are the base 10 digits of c_i if $c_i \neq 0$, and $c_i = +c_i^1 = +0$ if $c_i = 0$. This gives an injection from the set A of algebraic numbers to $\mathbb{N}$, so A is countable.

Chapter 7

Graph Theory

7.1. Graphs

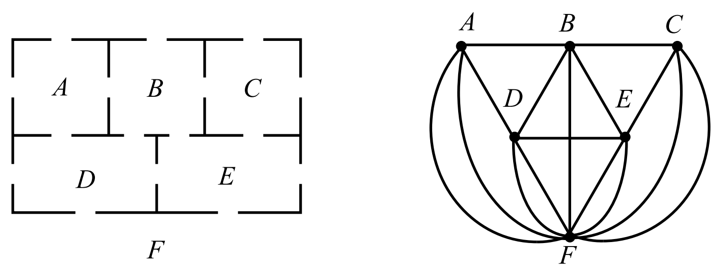
2. (a) $6^5 = 7776$
   (b) $15^7 = 170859375$
   (c) $\left(n + \binom{n}{2}\right)^k$. Any edge has either one endpoint (and there are n choices for the vertex at which such a loop may be based) or has two endpoints (and there are $\binom{n}{2}$ choices for the two end points). This gives $n + \binom{n}{2} = \frac{n^2+n}{2} = T_n$ ways to construct one edge, so there are $\left(n + \binom{n}{2}\right)^k = T_n^k$ ways to construct k edges on n vertices.
   (d) 0
   (e) 120
   (f) There are $\binom{n}{2}$ possible edges (with distinct endpoints) on n vertices, and we wish to choose k of them:

$$\binom{\binom{n}{2}}{k}.$$

6. (a) $e_1, e_2, e_1, e_6, e_{10}$, for example.
   (c) Impossible. If a walk has distinct vertices, it must have distinct edges, so every path is a trail.
   (e) e_7, e_8, e_{11}, e_{10}.

10. G is a connected graph if and only if $\bigcup_{i \in I} D_i$ is a connected subset of the plane. That is, G is a connected graph if and only if for every $a, b \in \bigcup_{i \in I} D_i$, there is a continuous curve contained in $\bigcup_{i \in I} D_i \subseteq \mathbb{R}^2$ from a to b.

13. By placing a doorway in each edge of the graph, the problem becomes analogous to those of Exercise 12. The associated graph is shown below. Since more than two vertices have odd degree (namely B, D, E, and F), the graph has no Eulerian trail, so it is impossible to draw a continuous curve bisecting each edge of the original graph.



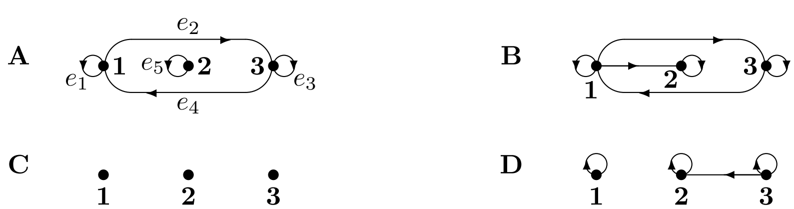
7.2. Matrices, Digraphs, and Relations

2.

(a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

8. (a)



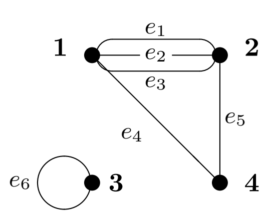
(b) Since the $(1,3)$ entry of A^3 is 4, there are four (v_1, v_3)-walks (i.e., $(1,3)$-walks) of length three. Referring to the edge labels in (a), they are $e_1e_1e_2, e_1e_2e_3, e_2e_3e_3$, and $e_2e_4e_2$.

17. (a)

$$\left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

(b) The adjacency matrix will be an $(m+n) \times (m+n)$ matrix containing an $m \times m$ square of zeros in the upper left corner, an $n \times n$ square of zeros in the lower right corner, and all other entries are ones.

21. (c)



	e_1	e_2	e_3	e_4	e_5	e_6
1	1	1	1	1	0	0
2	1	1	1	0	1	0
3	0	0	0	0	0	2
4	0	0	0	1	1	0

23. (c) Let $P = \prod_{i=1}^{n} \prod_{\substack{j\,=\,1 \\ j\,\neq\, i}}^{n} (1 - a_{ij}a_{ji})$. Then

$$
\begin{aligned}
P \neq 0 \quad &\iff \quad 1 - a_{ij}a_{ji} \neq 0 \;\; \forall i, j \in \{1, 2, \ldots n\}, i \neq j \\
&\iff \quad a_{ij}a_{ji} = 0 \forall i, j \in \{1, 2, \ldots n\}, i \neq j \\
&\iff \quad a_{ij} = 0 \text{ or } a_{ji} = 0 \;\; \forall i, j \in \{1, 2, \ldots n\}, i \neq j \\
&\iff \quad [a_{ij} = 1 \text{ and } a_{ji} = 1] \text{ implies } i = j \;\; \forall i, j \in \{1, \ldots n\} \\
&\iff \quad iRj \text{ and } jRi \text{ imply } i = j \;\; \forall i, j \in \{1, 2, \ldots n\} \\
&\iff \quad R \text{ is antisymmetric.}
\end{aligned}
$$

7.3. Shortest Paths in Weighted Graphs

6. (b) $agbhfe$ is a shortest (a, e)-path, and thus $bhfe$ must be a shortest (b, e)-path (for if there were a shorter (b, e)-path, appending it to agb would give a shorter (a, e)-path, contrary to $agbhfe$ being a shortest (a, e)-path).

8. (a) The shortest (d, v)-paths found by the implementation of Dijkstra's algorithm below are $dcba$, dcb, dc, d, $dgfe$, dgf, dg.

v	a	b	c	d	e	f	g
	∞_d	∞_d	1_d	$\mathbf{0}_{\boldsymbol{d}}$	∞_d	∞_d	2_d
	∞_d	5_c	$\mathbf{1}_{\boldsymbol{d}}$	$\mathbf{0}_{\boldsymbol{d}}$	∞_d	4_c	2_d
	∞_d	5_c	$\mathbf{1}_{\boldsymbol{d}}$	$\mathbf{0}_{\boldsymbol{d}}$	∞_d	3_g	$\mathbf{2}_{\boldsymbol{d}}$
	12_f	5_c	$\mathbf{1}_{\boldsymbol{d}}$	$\mathbf{0}_{\boldsymbol{d}}$	9_f	$\mathbf{3}_{\boldsymbol{g}}$	$\mathbf{2}_{\boldsymbol{d}}$
	8_b	$\mathbf{5}_{\boldsymbol{c}}$	$\mathbf{1}_{\boldsymbol{d}}$	$\mathbf{0}_{\boldsymbol{d}}$	9_f	$\mathbf{3}_{\boldsymbol{g}}$	$\mathbf{2}_{\boldsymbol{d}}$
	$\mathbf{8}_{\boldsymbol{b}}$	$\mathbf{5}_{\boldsymbol{c}}$	$\mathbf{1}_{\boldsymbol{d}}$	$\mathbf{0}_{\boldsymbol{d}}$	9_f	$\mathbf{3}_{\boldsymbol{g}}$	$\mathbf{2}_{\boldsymbol{d}}$
	$\mathbf{8}_{\boldsymbol{b}}$	$\mathbf{5}_{\boldsymbol{c}}$	$\mathbf{1}_{\boldsymbol{d}}$	$\mathbf{0}_{\boldsymbol{d}}$	$\mathbf{9}_{\boldsymbol{f}}$	$\mathbf{3}_{\boldsymbol{g}}$	$\mathbf{2}_{\boldsymbol{d}}$

9. (c) The shortest (f, v)-paths found by the implementation of Dijkstra's algorithm below are fa, fb, fgc, fgd, fbe, f, fg $fgch$.

v	a	b	c	d	e	f	g	h
	4_f	2_f	∞_f	∞_f	5_f	$\mathbf{0}_{\boldsymbol{f}}$	2_f	∞_f
	4_f	$\mathbf{2}_{\boldsymbol{f}}$	10_b	∞_f	4_b	$\mathbf{0}_{\boldsymbol{f}}$	2_f	∞_f
	4_f	$\mathbf{2}_{\boldsymbol{f}}$	7_g	8_g	4_b	$\mathbf{0}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	11_g
	$\mathbf{4}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	7_g	8_g	4_b	$\mathbf{0}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	11_g
	$\mathbf{4}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	7_g	8_g	$\mathbf{4}_{\boldsymbol{b}}$	$\mathbf{0}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	11_g
	$\mathbf{4}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	$\mathbf{7}_{\boldsymbol{g}}$	8_g	$\mathbf{4}_{\boldsymbol{b}}$	$\mathbf{0}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	10_c
	$\mathbf{4}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	$\mathbf{7}_{\boldsymbol{g}}$	$\mathbf{8}_{\boldsymbol{g}}$	$\mathbf{4}_{\boldsymbol{b}}$	$\mathbf{0}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	10_c
	$\mathbf{4}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	$\mathbf{7}_{\boldsymbol{g}}$	$\mathbf{8}_{\boldsymbol{g}}$	$\mathbf{4}_{\boldsymbol{b}}$	$\mathbf{0}_{\boldsymbol{f}}$	$\mathbf{2}_{\boldsymbol{f}}$	$\mathbf{10}_{\boldsymbol{c}}$

10. (a) The shortest (a, v)-paths found by the implementation of Dijkstra's algorithm below are a, ab, ac, $aefd$, ae, aef.

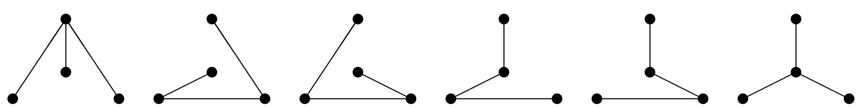
v	a	b	c	d	e	f
	$\mathbf{0}_{\boldsymbol{a}}$	4_a	8_a	∞_a	6_a	∞_d
	$\mathbf{0}_{\boldsymbol{a}}$	$\mathbf{4}_{\boldsymbol{a}}$	8_a	∞_a	6_a	∞_d
	$\mathbf{0}_{\boldsymbol{a}}$	$\mathbf{4}_{\boldsymbol{a}}$	8_a	∞_a	$\mathbf{6}_{\boldsymbol{a}}$	12_e
	$\mathbf{0}_{\boldsymbol{a}}$	$\mathbf{4}_{\boldsymbol{a}}$	$\mathbf{8}_{\boldsymbol{a}}$	∞_a	$\mathbf{6}_{\boldsymbol{a}}$	12_e
	$\mathbf{0}_{\boldsymbol{a}}$	$\mathbf{4}_{\boldsymbol{a}}$	$\mathbf{8}_{\boldsymbol{a}}$	17_f	$\mathbf{6}_{\boldsymbol{a}}$	$\mathbf{12}_{\boldsymbol{e}}$
	$\mathbf{0}_{\boldsymbol{a}}$	$\mathbf{4}_{\boldsymbol{a}}$	$\mathbf{8}_{\boldsymbol{a}}$	$\mathbf{17}_{\boldsymbol{f}}$	$\mathbf{6}_{\boldsymbol{a}}$	$\mathbf{12}_{\boldsymbol{e}}$

11. (b) The shortest (c, v)-paths found by the implementation of Dijkstra's algorithm below are ca, cab, c, cd, $cgfe$, cgf, cg, cdh.

v	a	b	c	d	e	f	g	h
	2_c	8_c	$\mathbf{0}_{\boldsymbol{c}}$	6_c	∞_c	∞_c	5_c	∞_c
	$\mathbf{2}_{\boldsymbol{c}}$	5_a	$\mathbf{0}_{\boldsymbol{c}}$	6_c	∞_c	∞_c	5_c	∞_c
	$\mathbf{2}_{\boldsymbol{c}}$	$\mathbf{5}_{\boldsymbol{a}}$	$\mathbf{0}_{\boldsymbol{c}}$	6_c	19_b	15_b	5_c	∞_c
	$\mathbf{2}_{\boldsymbol{c}}$	$\mathbf{5}_{\boldsymbol{a}}$	$\mathbf{0}_{\boldsymbol{c}}$	6_c	19_b	9_g	$\mathbf{5}_{\boldsymbol{c}}$	9_g
	$\mathbf{2}_{\boldsymbol{c}}$	$\mathbf{5}_{\boldsymbol{a}}$	$\mathbf{0}_{\boldsymbol{c}}$	$\mathbf{6}_{\boldsymbol{c}}$	19_b	9_g	$\mathbf{5}_{\boldsymbol{c}}$	7_d
	$\mathbf{2}_{\boldsymbol{c}}$	$\mathbf{5}_{\boldsymbol{a}}$	$\mathbf{0}_{\boldsymbol{c}}$	$\mathbf{6}_{\boldsymbol{c}}$	19_g	9_g	$\mathbf{5}_{\boldsymbol{c}}$	$\mathbf{7}_{\boldsymbol{d}}$
	$\mathbf{2}_{\boldsymbol{c}}$	$\mathbf{5}_{\boldsymbol{a}}$	$\mathbf{0}_{\boldsymbol{c}}$	$\mathbf{6}_{\boldsymbol{c}}$	12_f	$\mathbf{9}_{\boldsymbol{g}}$	$\mathbf{5}_{\boldsymbol{c}}$	$\mathbf{7}_{\boldsymbol{d}}$
	$\mathbf{2}_{\boldsymbol{c}}$	$\mathbf{5}_{\boldsymbol{a}}$	$\mathbf{0}_{\boldsymbol{c}}$	$\mathbf{6}_{\boldsymbol{c}}$	$\mathbf{12}_{\boldsymbol{f}}$	$\mathbf{9}_{\boldsymbol{g}}$	$\mathbf{5}_{\boldsymbol{c}}$	$\mathbf{7}_{\boldsymbol{d}}$

7.4. Trees

7. (a) Each of the first five spanning trees shown below can be rotated 60° or 120° to obtain other spanning trees. These $5 \times 3 = 15$ spanning trees and the sixth spanning tree shown below give 16 spanning trees.



(b) 100%. Classify the edges as "spokes" (the edges of the sixth spanning tree shown above) and "rim edges". By the Pigeonhole principle, three edges selected at random must contain at least two spokes or at least two rim edges. In either case, these two edges form a connected subgraph which contains 3 of the 4 vertices of G. The third edge has two endpoints, and one of them must be already among the three vertices incident on the first two edges. It follows that any three edges selected at random form a connected subgraph.

10. Using the Pythagorean theorem to find the lengths BE and AE, we find the lengths of the edges, in increasing order, are as shown:

BD 3, CE 3, DE 4, BC 4, BE 5, AC unknown, AB 12, AE 13.

Applying Kruskal's algorithm, we select edges BD, CE, either DE or BC, and AC. Since the side AC of unknown length does appear in the minimal spanning tree, we must compute its length to find the length of the minimal spanning tree. Since angles BEC and ABC are both complements of angle CBE, they have the same measure θ. From the 3-4-5 triangle, we see that $\cos\theta = \frac{3}{5}$. Applying the law of cosines to triangle ABC, we have $AC^2 =$

$4^2 + 12^2 - 2(4)(12)\frac{3}{5} = 102.4$, so $AC = \sqrt{102.4} \approx 10.1193$. Thus, the length of the minimal spanning tree is approximately $3 + 3 + 4 + 10.1193 = 20.1193$.

12. The weights of the edges of graph Q are given below in increasing order.

FG 5	BD 6	EH 7	FH 7	AB 8	EF 8	DE 9	CF 9
CG 9	AD 10	CD 10	GH 10	BE 11	DH 11	BC 12	CH 12

We go through the list and select the following edges to form a minimal spanning tree for Q: FG, BD, EH, FH, AB, DE, and CF. The only other edges forming a minimal spanning tree for G are FG, BD, EH, FH, AB, DE, and CG. The weight of the minimal spanning trees is 51.

The weights of the edges of graph R are given below in increasing order.

JK 5	FK 7	HL 8	BC 9	EF 9	EJ 9	IM 10	BG 11
FG 11	CD 12	HI 13	DH 15	AE 16	HM 17	GL 18	CG 19
DM 20	AB 21	KL 22	AG 24				

We go through the list and select the following edges to form a minimal spanning tree for R: JK, FK, HL, BC, EF, IM, BG, FG, CD, HI, DH, AE. The only other minimal spanning tree is obtained by replacing EF above by EJ. The weight of the minimal spanning trees is 126.

14. (a) The algorithm for maximal spanning trees is this: Start with any edge of maximal weight. From the remaining edges, add any edge of maximal weight which does not create a cycle. Repeat until all vertices are used. The result is a maximal spanning tree.

We now prove that the algorithm works. Given a connected weighted graph $G(V, E)$ with weight function $w : E \to [0, \infty)$. Let $m - 1 = \max\{w(e) | e \in E\}$ be the maximum weight in G, and define G' to be the graph $G'(V, E)$ having the same vertices and edges, but with the new weight function $w'(e) = m - w(e)$. Note that a list of the edges of G in increasing order of weights gives a list of the edges of G' in decreasing order of weights. Let $\mathcal{S}$ be the set of all spanning trees for $G(V, E)$. Then $\mathcal{S}$ is also the set of all spanning trees for $G'(V, E)$. Each $T \in \mathcal{S}$ has $v - 1$ edges where $v = |V|$. If $T = \{e_1, \ldots, e_{v-1}\} \in \mathcal{S}$, then $w(T) = w(e_1) + \cdots + w(e_{v-1})$ and $w'(T) = (m - w(e_1)) + \cdots + (m - w(e_{v-1})) = (v - 1)m - w(T)$. Thus, for $T \in \mathcal{S}$, $w(T)$ is maximum when $w'(T)$ is minimum, and conversely. This shows that a minimal spanning tree for G' is a maximal spanning tree for G, and conversely.

(b) As in the solution to Exercise 11, we list the edges of the graph of Exercise 11 in order.

AF 315	HK 320	CD 330	AB 332	BF 340	GK 345	FJ 350	BC 360
CF 360	JK 365	DG 370	FH 375	EI 375	IJ 378	AE 380	EF 380

Since we want a maximal spanning tree, we proceed greedily through the edges from the heaviest backwards through the list to the lightest, including edges as long as they do not create a cycle. The edges required are: EF, AE, IJ, EI, FH, DG, JK, CF, BC, GK. These edges form a maximal spanning tree, and the weight of this tree is $380 + 380 + 378 + 375 + 375 + 370 + 365 + 360 + 360 + 345 = 3688$.

Sequences

8.1. Sequences

5. $(151, 144, 137, 130, 123, 116, 109, 102, 95, 88, 81, 74, 67, 60, 53, 46, 39, 32, 25, 18, 11, 4)$. There are 22 nonnegative terms in this sequence, indicating that 22 is the largest number of sevens which can be subtracted from 158 so that the remaining difference (namely, 4) is nonnegative. This tells us that $158 \div 7$ gives a quotient of 22 with a remainder of 4.

6. (e) $(b_n)_{n=1}^{\infty} = (|3n-4|-5)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. ($f : \mathbb{R} \to \mathbb{R}$ is not increasing, but $f : \mathbb{N} \to \mathbb{N}$ is.)

11. (b) Suppose $b_n = b + ns$ and $a_n = ar^n$ for $n = 0, 1, 2, \ldots$. Then $a_{b_n} = ar^{b+ns} = ar^b \cdot r^{ns} = ar^b(r^s)^n$, so $(a_{b_n})_{n=0}^{\infty}$ is geometric with first term ar^b and ratio r^s.

12. (a) No. For example, if $(a_n)_{n=1}^{\infty} = (2, 4, 6, 8, 10, \ldots)$ and $(b_n)_{n=1}^{\infty} = (1, 2, 4, 8, \ldots)$, then $(a_{b_n})_{n=0}^{\infty} = (a_1, a_2, a_4, a_8, \ldots) = (2, 4, 8, 16, \ldots)$ which is not arithmetic.
    (b) No. For example, if $(a_n)_{n=1}^{\infty} = (b_n)_{n=1}^{\infty} = (1, 2, 4, 8, \ldots)$, then $(a_{b_n})_{n=0}^{\infty} = (a_1, a_2, a_4, \ldots) = (1, 2, 8, \ldots)$ which is not geometric.

16. (a) If D is a nonempty countable set, then either $D = \{d_1, d_2, \ldots, d_n\}$ is finite or D is countably infinite. If D is finite, then

$$(d_1, d_2, \ldots, d_{n-1}, d_n, d_n, d_n, d_n, d_n, \ldots)$$

is a sequence whose set of terms is D. If D is countably infinite, then there exists a bijection f from $\mathbb{N}$ to D, and $(f(n))_{n=1}^{\infty}$ is a sequence whose set of terms is D.
    (b) If $D = \{d_1, d_2, \ldots, d_n\}$, then any sequence in D is a subsequence of

$$(d_1, d_2, \ldots, d_n, d_1, d_2, \ldots, d_n, d_1, d_2, \ldots, d_n, \ldots).$$

If $D = \{d_1, d_2, d_3, \ldots)$ is countably infinite, then any sequence in D is a subsequence of

$$(d_1,\ d_1, d_2,\ d_1, d_2, d_3,\ d_1, d_2, d_3, d_4,\ d_1, d_2, d_3, d_4, d_5,\ d_1, d_2, \ldots).$$

8.2. Finite Differences

1. (a) The sequence of second differences is constantly 4. This tells us that the sequence is generated by a second degree polynomial $p(n) = an^2 + bn + c$. Since the second differences of the sequence determined by $p(n)$ are constantly $2!a = 4$, we find that $a = 2$. Since the first term 4 is $p(0) = c$, we have $p(n) = 2n^2 + bn + 4$. Now $p(1) = 3 = 2(1^2) + b(1) + 4$ implies $b = -3$, so $p(n) = 2n^2 - 3n + 4$.
   (c) The sequence of third differences is constantly 18. This tells us that the sequence is generated by a third degree polynomial $p(n) = an^3 + bn^2 + cn + d$. Since the third differences of the sequence determined by $p(n)$ are constantly $3!a = 18$, we find that $a = 3$. Since the first term -1 is $p(0) = d$, we have $p(n) = 3n^3 + bn^2 + cn - 1$. The equations $p(1) = 2$ and $p(2) = 23$ yield, respectively, $b + c = 0$ and $2b + c = 0$, and the only simultaneous solution to these equations is $b = c = 0$. Thus, $p(n) = 3n^3 - 1$.
   (e) The sequence of third differences is constantly $30 = 5 \cdot 3!$ and the initial term is 9, so the sequence is generated by a third degree polynomial of form $5n^3 + bn^2 + cn + 9$. The equations $p(1) = 11$ and $p(2) = 43$ yield $b + c = -3$ and $2b + c = -3$, giving $b = 0$ and $c = -3$. Thus, $p(n) = 5n^3 - 3n + 9$.
   (g) The sequence of fourth differences is constantly $48 = 2 \cdot 4!$ and the initial term is 38, so the sequence is generated by a polynomial of form $p(n) = 2n^4 + bn^3 + cn^2 + dn + 38$. The equations $p(1) = 40, p(2) = 70$, and $p(3) = 200$ yield $b + c + d = 0 = 4b + 2c + d = 9b + 3c + d$. Clearly $b = c = d = 0$ is a solution, so $p(n) = 2n^4 + 38$.

3. (c) $a_n = f(n) = n^2 + 2^n$. Observe that the 3^{rd} differences (and all m^{th} differences for $m \geq 3$) are $1, 2, 4, 8, \ldots$. The sequence $(2^n)_{n=0}^{\infty}$ has $1, 2, 4, 8, \ldots$ as m^{th} differences for all natural numbers m. The fact that the first and second differences of our sequence are not $1, 2, 4, 8, \ldots$ suggests that the terms of our sequence are $2^n + p(n)$ where $p(n)$ is a second degree polynomial. (The addition of such a polynomial will alter only the first and second differences, since all subsequent differences of $p(n)$ would be zero.) Subtracting 2^n from the n^{th} term of the original sequence leaves the sequence n^2, so the original sequence is given by $a_n = n^2 + 2^n$.

5. If $a_n = f(n) = \sum_{i=0}^{n} i^2$, then the sequence of first differences is $(0^2, 1^2, 2^2, 3^2, \ldots)$ and thus the sequence of third differences is constantly $2! = 2$. Thus, $a_n = f(n) = an^3 + bn^2 + cn + d$, and since the third differences of this sequence are $3!a = 2$, we have $a = \frac{2}{3!} = \frac{1}{3}$. Since $f(0) = 0 = d$, we now have $f(n) = \frac{1}{3}n^3 + bn^2 + cn$. From $f(1) = 1^2 = \frac{1}{3} + b + c$ and

$f(2) = 1^2 + 2^2 = 5 = \frac{8}{3} + 4b + 2c$, we find that $b = \frac{1}{2}$ and $c = \frac{1}{6}$, so

$$f(n) = 1^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n}{6}(2n^2 + 3n + 1) = \frac{n(n+1)(2n+1)}{6}.$$

7. If the k^{th} differences of $(a_i)_{i=1}^{\infty}$ are generated by an n^{th} degree polynomial, then the n^{th} differences of the k^{th} differences of of $(a_i)_{i=1}^{\infty}$ are a nonzero constant. Thus, the $(n+k)^{th}$ differences of $(a_i)_{i=1}^{\infty}$ are constant and nonzero, so $(a_i)_{i=1}^{\infty}$ is generated by an $(n+k)^{th}$ degree polynomial.

12. (b) The first differences of $-5, -2, 4, 16, 40, \ldots$ agree with those of $(3 \cdot 2^i)_{i=0}^{\infty} = (3, 6, 12, 24, 48, \ldots)$, so these two sequences differ by a constant. The original sequence is $(3 \cdot 2^i - 8)_{i=0}^{\infty}$.

8.3. Limits of Sequences of Real Numbers

3. Given $\epsilon > 0$, we wish to find $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \left| \frac{1-n^2}{3n^2+1} - \frac{-1}{3} \right| < \epsilon.$$

But

$$\left| \frac{1-n^2}{3n^2+1} - \frac{-1}{3} \right| = \left| \frac{3-3n^2}{3(3n^2+1)} + \frac{3n^2+1}{3(3n^2+1)} \right| = \left| \frac{4}{3(3n^2+1)} \right| = \frac{4}{9n^2+3}.$$

Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.

$$\begin{aligned} n \geq N \quad &\Rightarrow \quad 9n^2 + 3 \geq 4N \\ &\Rightarrow \quad \frac{4}{9n^2+3} \leq \frac{4}{4N} = \frac{1}{N} < \epsilon. \end{aligned}$$

Thus, $n \geq N \Rightarrow \left| \frac{1-n^2}{3n^2+1} - \frac{-1}{3} \right| < \epsilon$, as needed.

[Or, choose $N \in \mathbb{N}$ such that $\frac{4}{9N^2+3} < \epsilon$, if you believe such an N exists.]

5. Given $M < 0$, we we wish to find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \frac{2n^2+1}{3-n} < M$. Choose $N = \max\{4, -M\}$ and suppose $n \geq N$. Now $n \geq N \Rightarrow n \geq 4$, which implies

$$\begin{aligned} \frac{2n^2+1}{3-n} &= \frac{2n^2}{3-n} + \frac{1}{3-n} \\ &< \frac{2n^2}{3-n} = \frac{-n(-2n)}{3-n} = -n\left(\frac{2n}{n-3}\right) = -n\left(\frac{n+n}{n-3}\right) \\ &< -n \qquad (\text{since } \frac{n+n}{n-3} > 1). \end{aligned}$$

Now because $-n \leq -N \leq M$, we have $n \geq N$ now implies $\frac{2n^2+1}{3-n} < M$, as needed.

10. The functions that preserve all limits are known as *continuous functions*. Our example will necessarily be discontinuous. Let $f(x) = 1$ if $x \neq 0$ and $f(0) = 0$. Let $(a_n)_{n=1}^{\infty} = (\frac{1}{n})_{n=1}^{\infty}$. Now $f(a_n) = f(\frac{1}{n}) = 1$ for any $n \in \mathbb{N}$, so $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 1 = 1$, but $f(\lim_{n\to\infty} a_n) = f(\lim_{n\to\infty} \frac{1}{n}) = f(0) = 0 \neq 1 = \lim_{n\to\infty} f(a_n)$.

15. (b) The following statements are equivalent:

(i) $\lim_{n\to\infty} a_n = \infty$
(ii) $\forall M > 0\ \exists N \in \mathbb{N}, N > 100$ such that $n \geq N \Rightarrow a_n > M$
(iii) $\forall M > 0\ \exists N \in \mathbb{N}, N > 100$ such that $n + 100 \geq N \Rightarrow a_{n+100} > M$
(iv) $\forall M > 0\ \exists N' = N - 100 \in \mathbb{N}$ such that $n \geq N' \Rightarrow b_n > M$
(v) $\lim_{n\to\infty} b_n = \infty$

18. Suppose $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Suppose $\epsilon > 0$ is given. Then there exist $N_a, N_b \in \mathbb{N}$ such that $n \geq N_a \Rightarrow |a_n - A| < \frac{\epsilon}{2}$ and $n \geq N_b \Rightarrow |b_n - B| < \frac{\epsilon}{2}$. Now for $n \geq \max\{N_a, N_b\}$, we have

$$\begin{aligned} |A_n + b_n - (A+B)| &= |a_n - A + b_n - B| \\ &\leq |a_n - A| + |b_n - B| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty}(a_n + b_n) = A + B = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

8.4. Some Convergence Properties

1. (a) False. Consider $a_n = \frac{1}{n}, b_n = \frac{1}{2n}$.
(b) True. If $\lim_{n\to\infty} a_n = A < B = \lim_{n\to\infty} b_n$, take $\epsilon = \frac{B-A}{2}$. Now there exists $M_a, M_b \in \mathbb{N}$ such that

$$A - \epsilon < a_n < A + \epsilon \leq B - \epsilon < b_j < B + \epsilon$$

for any $n \geq M_a$ and any $j \geq M_b$. Now for $M = \max\{M_a, M_b\}$, we have $a_n < b_n\ \ \forall n \geq M$.

6. Any decreasing sequence $(a_n)_{n=1}^{\infty}$ which is not bounded below by any M must diverge to $-\infty$, for given $M < 0$, $\exists N \in \mathbb{N}$ such that $a_N < M$, and therefore $a_n \leq M\ \ \forall n \geq N$.

If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of real numbers which is bounded below, then $(-a_n)_{n=1}^{\infty}$ is an increasing sequence of real numbers bounded above, and therefore $(-a_n)_{n=1}^{\infty}$ converges to a limit $-L$ by the proof of Theorem 8.4.1. It follows that $(a_n)_{n=1}^{\infty}$ converges to L.

Thus, any decreasing sequence of real numbers either converges or diverges to $-\infty$.

10. (a) Dividing the numerator and denominator of the expression for a_n by n^2 gives

$$a_n = \frac{1 - \frac{100}{n}}{1 + \frac{2}{n^2}} = \frac{p(\frac{1}{n})}{q(\frac{1}{n})} \quad \text{where } p(x) = 1 - 100x \ \text{ and } \ q(x) = 1 + 2x^2.$$

Similarly, we find $r(x) = 1 + 100x$ and $s(x) = 1 + 2x^2$.
(b)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{p(\frac{1}{n})}{q(\frac{1}{n})} = \frac{\lim_{n\to\infty} p(\frac{1}{n})}{\lim_{n\to\infty} q(\frac{1}{n})} = \frac{p(0)}{q(0)} = \frac{1}{1} = 1,$$

and similarly, $\lim_{n\to\infty} b_n = 1$.
(c) Since $a_n \leq c_n \leq b_n\ \ \forall n \in \mathbb{N}$ and the outer two sequences converge to 1 as $n \to \infty$, it follows that $\lim_{n\to\infty} c_n = 1$.

8.5. Infinite Arithmetic

2. $1 - 1 + 1 - 1 + 1 - \cdots$ diverges. The odd partial sums are all 1 and the even partial sums are all 0, so the sequence $(1, 0, 1, 0, 1, 0, \ldots)$ of partial sums does not converge.

5. $d_n = 0.\underbrace{111\cdots1}_{n \text{ digits}}$ and $\lim_{n\to\infty} d_n = 0.\overline{111} = 1$. (Just as $0.\overline{999} = 1.0$ in base 10, in base 2 we have $0.\overline{111} = 1.0$.)

12. (c) Let $P_n = \prod_{n=0}^{\infty}\left(1 + \frac{1}{r^{(2^n)}}\right)$ and $s_n = \sum_{j=0}^{\infty} r^j$. Note that

$$\begin{aligned} P_0 &= 1 + \frac{1}{r} = s_1 \\ P_1 &= \left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{r^2}\right) = 1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} = s_3 \\ P_2 &= P_1\left(1 + \frac{1}{r^4}\right) = s_7 \\ P_3 &= s_{15} \end{aligned}$$

and in general, $P_n = s_{2^{n+1}-1}$.
Now if $r \in (0,1)$, then $\sum_{j=0}^{\infty} r^j$ converges, and this implies the convergence of $(s_n)_{n=0}^{\infty}$ and thus $(s_{2^{n+1}-1})_{n=0}^{\infty} = (P_n)_{n=0}^{\infty}$. If $r \geq 1$, then $\sum_{j=0}^{\infty} r^{(2^j)}$ diverges since $\lim_{j\to\infty} r^{(2^j)} \neq 0$, and by Theorem 8.5.4, $\lim_{n\to\infty} P_n$ also diverges. Thus, for $r > 0$, $\sum_{j=0}^{\infty} r^j = \prod_{j=0}^{\infty}\left(1 + \frac{1}{r^{(2^j)}}\right)$.

13. Let p_k be the k^{th} partial product.
    (b) $(p_{100k})_{k=1}^{10} = (0.6667326, 0.6666832, 0.6666740, 0.6666708, 0.6666693, 0.6666685, 0.6666680, 0.6666677, 0.6666674, 0.6666673)$. This suggests that the partial products decrease to $\frac{2}{3}$.

14. (b) As $n \to \infty$, the graphs of $f_n(x)$ converge to the graph of $y = \cos(x)$.

16. (a) Let $p_k = \sqrt{3 + \sqrt{2 + \cdots \sqrt{a_k}}}$ where $(a_i)_{i=1}^{\infty} = (3, 2, 3, 2, 3, 2, \ldots)$. Now $p_{k+2} = \sqrt{3 + \sqrt{2 + p_k}}$. Observe that $p_1 = \sqrt{3} < 3$ and $p_2 = \sqrt{3 + \sqrt{2}} < \sqrt{3 + 2} < \sqrt{9} = 3$. Now suppose $p_1, \ldots, p_{k+1} < 3$. Then $p_{k+2} = \sqrt{3 + \sqrt{2 + p_k}} < \sqrt{3 + \sqrt{2 + 3}}$ since $g(x) = \sqrt{3 + \sqrt{2 + x}}$ is an increasing function. Since $\sqrt{3 + \sqrt{5}} < \sqrt{3 + 5} < \sqrt{9} = 3$, we have $p_{k+2} < 3$. By mathematical induction, $(p_k)_{k=1}^{\infty}$ is bounded above by 3.

20. Any periodic sequence of nonnegative real numbers is bounded above, and thus the sequence of partial expressions for the associated infinite additive nested radical is increasing and bounded above, and hence is convergent.

21. (a) If $\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} = 3$, then $a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} = 9$, or $a + 3 = 9$, so $a = 6$.

8.6. Recurrence Relations

2. Given a k^{th}-order recurrence relation and k initial conditions $a_1, \ldots, a_k$, this uniquely determines a_{k+1}. Now suppose $a_{j-k+1}, a_{j-k+2}, \ldots, a_j$ have been

uniquely determined. The recurrence relation then gives a_{j+1}. By mathematical induction, we see that a_n is uniquely determined for any $n \in \mathbb{N}$, and thus $f(n) = a_n$ is the unique solution to the recurrence relation.

5. The Fibonacci sequence is given by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n\ \forall n \in \mathbb{N} \cup \{0\}$. The characteristic equation $r^2 = r + 1$ has roots $r = \frac{1 \pm \sqrt{5}}{2}$, which provide the basic solutions $((\frac{1+\sqrt{5}}{2})^n)_{n=0}^{\infty}$ and $((\frac{1-\sqrt{5}}{2})^n)_{n=0}^{\infty}$ to the recurrence relation. We wish to find a linear combination $c(\frac{1+\sqrt{5}}{2})^n + d(\frac{1-\sqrt{5}}{2})^n$ which satisfies the initial conditions:

$$\begin{aligned} 0 = F_0 &= c + d \qquad\qquad (\text{so } d = -c) \\ 1 = F_1 &= c\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right) \\ &= c\left(\frac{1+\sqrt{5}}{2}\right) - c\left(\frac{1-\sqrt{5}}{2}\right) \\ &= \sqrt{5}c \end{aligned}$$

It follows that $c = \frac{1}{\sqrt{5}}$ and $d = \frac{-1}{\sqrt{5}}$, so

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} \qquad \forall n \in \mathbb{N} \cup \{0\}.$$

6. (a) The recurrence relation $a_{n+2} = 3a_{n+1} + 10a_n$ has characteristic equation $r^2 - 3r - 10 = 0 = (r-5)(r+2)$, so the general solution to the recurrence relation is $a_n = b5^n + c(-2)^n$. The initial condition $a_0 = -2$ gives $-2 = b + c$ and the initial condition $a_1 = 11$ gives $11 = 5b - 2c$. These two linear equations in b and c have a unique solution $b = 1, c = -3$. Thus, $a_n = 1 \cdot 5^n - 3(-2)^n = 5^n - 3(-2)^n$.

   (c) The recurrence relation $a_{n+4} = 13a_{n+2} - 36a_n$ has characteristic equation $r^4 - 13r^2 + 36 = 0 = (r^2 - 9)(r^2 - 4)$, which has roots $\pm 3, \pm 2$, so the general solution to the recurrence relation is $a_n = b3^n + c(-3)^n + d2^n + e(-2)^n$. The initial conditions $a_0 = 14, a_1 = -5, a_2 = 101$ and $a_3 = -35$ give

$$\begin{aligned} b + c + d + e &= 14 \\ 3b - 3c + 2d - 2e &= -5 \\ 9b + 9c + 4d + 4e &= 101 \\ 27b - 27c + 8d - 8e &= -35 \end{aligned}$$

   This system may be solved using standard linear algebra techniques, or we may reduce this system of 4 equations in 4 unknowns to two systems of 2 equations in 2 unknowns: The first and third equations form a system in unknowns $(b + c)$ and $(d + e)$

$$\begin{aligned} (b+c) + (d+e) &= 14 \\ 9(b+c) + 4(d+e) &= 101 \end{aligned}$$

with solutions $b+c=9, d+e=5$. The second and fourth equations form a system in unknowns $(b-c)$ and $(d-e)$

$$\begin{aligned} 3(b-c)+2(d-e) &= -5 \\ 27(b-c)+8(d-e) &= -35 \end{aligned}$$

with solutions $b-c=-1, d-e=-1$. Now combining $b+c=9$ and $b-c=-1$, we find $b=4, c=5$, and combining $d+e=5$ and $d-e=-1$, we find $d=2, e=3$. Thus, the solution to the recurrence relation with the given initial conditions is $a_n = 4\cdot 3^n + 5(-3)^n + 2\cdot 2^n + 3(-2)^n$.

8. (a) The characteristic equation is $r^2 = 4r-4$ or $(r-2)^2 = 0$, so $r=2$ is a repeated root of multiplicity 2.
   (b) Substituting $a_n = 2^n$ into the recurrence relation, we get $2^{n+2} = 4\cdot 2^{n+1} - 4\cdot 2^n$, or upon dividing by 2^n, $2^2 = 4\cdot 2-4$, which is true. Substituting $a_n = n2^n$ into the recurrence relation, we get $(n+2)2^{n+2} = 4(n+1)2^{n+1} - 4n2^n$, or upon dividing by $4\cdot 2^n$, $(n+2) = (n+1)2-n$, which is true. Now by Theorem 8.6.2, $a_n = c2^n + dn2^n$ is a solution to the recurrence relation.
   (c) The initial conditions give $5 = a_0 = c+0d$ and $-4 = a_1 = 2c+2d$, so $c=5$ and $d=-7$, and thus $a_n = 5(2^n) - 7n(2^n) = 2^n(5-7n)\ \forall n \geq 0$.

Fibonacci Numbers and Pascal's Triangle

9.1. Pascal's Triangle

3. (a) Ways to write 4 as an ordered sum of natural numbers

using one term	4	1 way
using two terms	1+3 = 3+1 = 2+2	3 ways
using three terms	1+1+2 = 1+2+1 = 2+1+1	3 ways
using four terms	1+1+1+1	1 way

There are $8 = \sum_{j=0}^{3} \binom{3}{j}$ solutions. By the results of Section 4.4, the number of natural number solutions to $x_1 + \cdots + x_k = 4$ is the same as the number of whole number solutions to $x_1' + \cdots x_k' = 4 - k$, which will be $\binom{4-k+k-1}{4-k} = \binom{3}{4-k}$. Summing from $k = 1$ to 4 gives the number we wish, namely $\sum_{k=1}^{4} \binom{3}{4-k} = \sum_{j=0}^{3} \binom{3}{j} = 2^3 = 8$.

(b) The number of natural number solutions to $x_1 + \cdots + x_k = m$ is the same as the number of whole number solutions to $x_1' + \cdots x_k' = m - k$, and this number is $\binom{m-k+k-1}{m-k} = \binom{m-1}{m-k}$. Summing from $k = 1$ to m gives the number we wish, namely

$$\sum_{k=1}^{m} \binom{m-1}{m-k} = \sum_{j=0}^{m-1} \binom{m-1}{j} = 2^{m-1}.$$

The last equality holds from the result of Example 9.1.1.

5. Of the $2^4 = 1 + 4 + 6 + 4 + 1 = 16$ subsets of $\{a, b, c, d\}$, half of them ($2^3 = 1 + 6 + 1 = \binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 8$ of them) have an even number of elements. These subsets are $\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$, and $\{a, b, c, d\}$.

10. The seven entries around $\binom{n}{k}$ are

$$\begin{array}{ccccc} & \binom{n-1}{k-1} & & \binom{n-1}{k} & \\ \binom{n}{k-1} & & \binom{n}{k} & & \binom{n}{k+1} \\ & \binom{n+1}{k} & & \binom{n+1}{k+1} & \end{array} \qquad \text{which we label as} \quad \begin{array}{ccccc} & a & & b & \\ c & & d & & e \\ & f & & g & \end{array}.$$

Let $h = \binom{n+2}{k+1}$. We wish to show that $a+b+c+d+e+f+g = 2h$. Now

$$\begin{aligned} a+b+c+d+e+f+g &= [(a+b)+c]+(d+e)+(f+g) \\ &= [d+c]+(g)+(h) \\ &= f+g+h \\ &= 2h, \qquad \text{as needed.} \end{aligned}$$

13. (a)

$$\left(\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n}\right)\cdot(0,1,2,\ldots,n) = 2^{n-1}\cdot n.$$

14. (a)

$$\begin{aligned} &\left(\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n}\right)\cdot(0,1,2,\ldots,n) \\ &= \binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+(n-1)\binom{n}{n-1}+n\binom{n}{n} \\ &= n+\frac{n!}{(n-2)!1!}+\frac{n!}{(n-3)!2!}+\cdots+\frac{n!}{1!(n-2)!}+n \\ &= n\left[1+\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-2}+1\right] \\ &= n\cdot 2^{n-1} \text{ (by the result of Example 9.1.1).} \end{aligned}$$

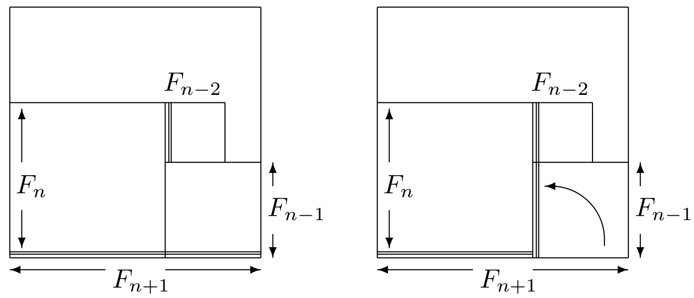
17. (a) Observe that row k contains only odd entries if and only if row $k+1$ contains only even entries except for the initial and final 1's. Thus, by Theorem 9.1.8, the rows which contain only odd entries are rows $2^m - 1$ for $m \in \mathbb{N} \cup \{0\}$.
    (b) The entries of row m alternate odd, even, odd, even, ... if and only if the entries of row $m+1$ are all odd, and by part (a), this occurs if and only if $m+1 = 2^n - 1$ $(n \in \mathbb{N})$, if and only if $m = 2^n - 2$ for some $n \in \mathbb{N}$.

9.2. The Fibonacci Numbers

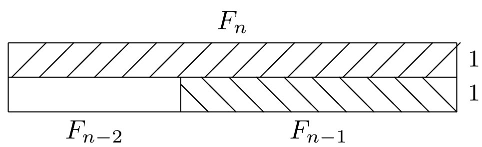
1. (a) $F_7 = 13$. See Example 9.2.2, and interpret $1''$ high blocks as \$5 payments and $2''$ high blocks as \$10 payments.
   (b) $F_{12} = 144$.
   (c) F_{n+1}.

5. (d)

$$F_{n+1} + F_{n-2} \quad = \quad 2F_n$$



Or, consider the following rectangle with area $2F_n$. The shaded region has area F_{n+1} (since $F_{n-1} + F_n = F_{n+1}$) but also has area $2F_n - F_{n-2}$.



6. $F_0 = 0 = (-1)^1 F_0$ and $F_{-1} = 1 = (-1)^2 F_1$. Now suppose $F_{-k} = (-1)^{k+1} F_k$ for $k = 0, 1, \ldots, j$. Now

$$\begin{aligned} F_{-(j+1)} &= F_{-(j-1)} - F_{-j} \\ &= (-1)^j F_{j-1} - (-1)^{-j+1} F_j \\ &= (-1)^{j+2}(F_{j-1} + F_j) \\ &= (-1)^{j+2} F_{j+1} \end{aligned}$$

By mathematical induction, $F_{-n} = (-1)^{n+1} F_n$ for any integer $n \geq 0$, and dividing by $(-1)^{n+1}$ shows that the result holds for all negative integers, as well.

8. $F_m^2 + F_{m-1}^2 = F_{2m-1}$. Apply Theorem 9.2.4 with $n = 2m - 1$ and $j = m - 1$.

10. After trying a few values of n, the formula is easily recognized to be

$$F_n F_{n+1} - F_{n-1} F_{n+2} = (-1)^{n+1}.$$

Dividing by $(-1)^{n+1}$ gives

$$(-1)^{n+1} F_n F_{n+1} + (-1)^n F_{n-1} F_{n+2} = 1.$$

Observing that $(-1)^{j+1} F_j = F_{-j}$, we have

$$F_{-n} F_{n+1} + F_{1-n} F_{n+2} = 1 = F_1 = F_2 = F_{-1}.$$

This formula looks very similar to one proved in Theorem 9.2.4:

$$F_{j+1} F_{m-j} + F_j F_{m-j-1} = F_m.$$

We would hope to find appropriate values of m and j which transform the result of Theorem 9.2.4 into the formula we wish to prove. Taking $m = 2$ and $j = -n$ gives the result.

12. (b) $(F_{n+1}^2 - F_{n-1}^2)_{n=2}^7 = (3, 8, 21, 55, 144, 377)$. The formula is $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$. Applying Theorem 9.2.4 with $m = j$ gives

$$\begin{aligned} F_{2j} &= F_{j+1}F_j + F_jF_{j-1} \\ &= F_j(F_{j+1} + F_{j-1}) \\ &= (F_{j+1} - F_{j-1})(F_{j+1} + F_{j-1}) \\ &= F_{j+1}^2 - F_{j-1}^2. \end{aligned}$$

9.3. The Golden Ratio

4. The sequence of partial expressions is

$$(\sqrt{1}, \sqrt{1 - \sqrt{1}}, \sqrt{1 - \sqrt{1 - \sqrt{1}}}, \ldots) = (1, 0, 1, 0, 1, 0, \ldots),$$

which diverges. [Were one not to notice this divergence, one would be tempted to say the value of the nested radical is x where $x = \sqrt{1 - x}$, so that $x = \frac{-1 \pm \sqrt{5}}{2}$. All this shows, however, is that *if* the nested radical converged, its value would be one of those given.]

6. Let $ABCD$, M, E, and F be as described and take $AB = 1$. Then $BC = 1$ and $MB = \frac{1}{2}$, so $CM = \sqrt{1^2 + (\frac{1}{2})^2} = \frac{\sqrt{5}}{2}$. Now $AE = AM + ME = AM + MC = \frac{1}{2} + \frac{\sqrt{5}}{2} = \varphi$, so $\frac{AE}{AD} = \frac{\varphi}{1} = \varphi$, and $AEFD$ is a golden rectangle.

7. The restrictions $lwh = 1$, $\sqrt{l^2 + w^2 + h^2} = 2$, and $h = 1$ give $l^2 + w^2 = 3$ and $l = \frac{1}{w}$. Substituting the latter equation into the former and multiplying through by w^2 gives $1 + w^4 = 3w^2$, a quadratic in w^2 with solutions

$$w^2 = \frac{3 + \sqrt{5}}{2} = 1 + \varphi = \varphi^2$$

and

$$w^2 = \frac{3 - \sqrt{5}}{2} = \frac{2}{3 + \sqrt{5}} = \frac{1}{\varphi^2}.$$

Since w must be positive, we have $w = \varphi$ and $l = \frac{1}{w} = \frac{1}{\varphi}$, or $w = \frac{1}{\varphi}$ and $l = \frac{1}{w} = \varphi$.

9.4. Fibonacci Numbers and the Golden Ratio

1. These problems use the fact that $\varphi^2 = \varphi + 1$, and (multiplying by φ^n) $\varphi^{n+2} = \varphi^{n+1} + \varphi^n$.

(b)

$$\begin{aligned} 2\varphi^4 - 3\varphi^2 - 8 &= 2(\varphi + 1)^2 - 3(\varphi + 1) - 8 \\ &= 2(\varphi^2 + 2\varphi + 1) - 3\varphi - 3 - 8 \\ &= 2(\varphi + 1) + 4\varphi + 2 - 3\varphi - 11 \\ &= 3\varphi - 7 \end{aligned}$$

(d)

$$\begin{aligned}2\varphi^5 - 3\varphi^4 + 1 &= 2(\varphi^4 + \varphi^3) - 3\varphi^4 + 1\\ &= -1\varphi^4 + 2\varphi^3 + 1\\ &= -(\varphi^3 + \varphi^2) + 2\varphi^3 + 1\\ &= \varphi^3 - \varphi^2 + 1\\ &= (\varphi^2 + \varphi) - \varphi^2 + 1\\ &= \varphi + 1\end{aligned}$$

6. (a) Yes. If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are additive sequences and $c_n = a_n + b_n$, then

$$\begin{aligned}c_{n+2} &= a_{n+2} + b_{n+2}\\ &= (a_{n+1} + a_n) + (b_{n+1} + b_n)\\ &= (a_{n+1} + b_{n+1}) + (a_n + b_n)\\ &= c_{n+1} + c_n,\end{aligned}$$

so $(c_n)_{n=1}^{\infty}$ is additive as well.

(b) (i) $(F_{n-1} + F_{n+1})_{n=1}^{\infty} = (1, 3, 4, 7, 11, \ldots) = (L_n)_{n=1}^{\infty}$

(ii) $(L_{n-1} + L_{n+1})_{n=2}^{\infty} = (5, 10, 15, 25, 40, \ldots) = (5F_n)_{n=2}^{\infty}$

(iii) $(\varphi^n + (\varphi')^n)_{n=1}^{\infty} = (1, 3, 4, 7, 11, \ldots) = (L_n)_{n=1}^{\infty}$

(iv) $(\frac{F_{2n}}{F_n})_{n=1}^{\infty} = (1, 3, 4, 7, 11, \ldots) = (L_n)_{n=1}^{\infty}$

7. (a) From Exercise 4 (a), we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{F_{n-1}a_1 + F_n a_2}{F_{n-2}a_1 + F_{n-1}a_2}\\ &= \lim_{n\to\infty} \frac{F_{n-1}(a_1 + \frac{F_n}{F_{n-1}}a_2)}{F_{n-2}(a_1 + \frac{F_{n-1}}{F_{n-2}}a_2)}\\ &= \varphi\left(\frac{a_1 + \varphi a_2}{a_1 + \varphi a_2}\right)\\ &= \varphi.\end{aligned}$$

11. $$\frac{x}{1 + x - x^2} = 1x - 1x^2 + 2x^3 - 3x^4 + 5x^5 - \cdots + (-1)^{n+1}F_n x^n + \cdots$$

9.5. Pascal's Triangle and the Fibonacci Numbers

2. (c)

Number of \$50 calculators	Number of \$100 calculators	Number of ways to distribute
8	0	$\binom{15}{8} = 6435$
6	1	$\binom{15}{6}\binom{9}{1} = 45045$
4	2	$\binom{15}{4}\binom{11}{2} = 75075$
2	3	$\binom{15}{2}\binom{13}{3} = 30030$
0	4	$\binom{15}{4} = 1365$
		157,950

The total number of outcomes is 157, 950, which is not a Fibonacci number. It falls between $F_{26} = 121,393$ and $F_{27} = 196,418$.

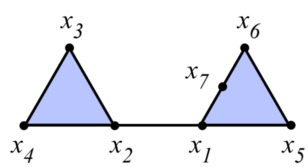
5. The formula follows immediately from Theorem 9.5.2 and the fact that $\binom{m}{j} = \binom{m}{m-j}$. This new formula would have been suggested by Example 9.5.1 if the right column of the tables there had listed the number of ways to distribute \$50 calculators (rather than the \$100 ones) among those receiving new calculators.

8. (a) $5^n F_{2n}$
   (b) $5^n F_{2n+1}$
   (c) $5^{n-1} L_{2n}$
   (d) $5^n F_{2n+3}$.

Chapter 10

Combinatorial Geometry

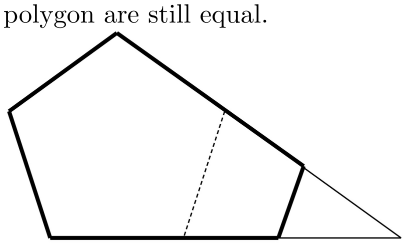
10.1. Polygons and Convex Sets



1.
   (a) $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1$ are sides of a non-degenerate polygon.
   (b) $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_1$ are sides of a degenerate polygon. The consecutive vertices x_6, x_7, x_1 are collinear.
   (c) $x_2x_3, x_3x_4, x_4x_1, x_1x_5, x_5x_6, x_6x_1, x_1x_2$ are not sides of a polygon since the vertices $(x_2, x_3, x_4, x_1, x_5, x_6, x_1, x_2)$ are not distinct. x_1 is repeated.

6. Prove that for any $n \geq 4$, there exists a non-regular n-gon with all n internal angles of the same measure.

   The regular n-gon exists, since we may mark the angles $2\pi k/n$ on the unit circle and connect the dots. If $n = 4$, take a 3×4 rectangle. If $n > 4$, consider four consecutive vertices a, b, c, d of a regular n-gon. Now ab extended must intersect dc extended at a point p outside the regular n-gon. Let b' be the midpoint of bp and c' be the midpoint of cp. Now replace sides ab, bc, cd by $ab', b'c', c'd$. Since $c'd'$ is parallel to cd, all the interior angles of the new polygon are still equal.



12. Suppose P is a simple polygon having sides $s_1, \ldots, s_n$ which lie on lines $l_1, \ldots, l_n$. Show that P is convex if and only if for each $i = 1, \ldots, n$, P lies entirely on one side of l_i.

Suppose P is convex, l_i is a side line, and $a, b \in P$ lie on opposite sides of l_i. Assume further that the polygon is non-degenerate, so no pair of adjacent sides are collinear. By convexity, if s_i lies on l_i, no other points on l_i are in P (or else P could contain a side which strictly contains s_i). The hypotheses on a and b imply that segement ab intersects l_i and thus intersects s_i. Now since a, b are on opposite sides of l_i, neither is on s_i. For each $x \in s_i$, the segments ax and xb must lie in P by convexity. The union of all such segments ax and xb covers a open disk around the midpoint y of ab. This is a contradiction since y is on s_i, and a side of a simple polygon cannot have interior points on both sides of it.

Conversely, suppose that for every line l_i containing a side s_i of P, P lies entirely on one side of l_i. If P is not convex, there exist $a, b \in P$ and c on the segment ab with $c \notin P$. Without loss of generality, we may assume a and b are interior points of P: $c \notin P$ implies there is a small circular disk $B(c, \epsilon)$ of radius $\epsilon > 0$ centered at c which does not intersect P. $a, b \in P$ implies there exists a', b' in the interior of P which are arbitrarily close to a and b, respectively. With a', b' close enough to a and b, the segment $a'b'$ must intersect the disk $B(c, \epsilon)$ at some point $c' \notin P$, and replacing a, b, c by a', b', c' shows that we may assume a, b are in the interior of P. In a similar manner, fluctuating the endpoints if needed, we may stipulate without loss of generality that segment ab is not parallel to any of the sides $s_1, s_2, \ldots, s_n$. Now there must exist a'' in the open segment ab with $a'' \in P$ such that the open segment $a''c$ does not intersect P. (That is, a'' is the "last" point of P on the segment ac.) With its position as the last point in P along the direction of ab, a'' must lie on a side s_i of P. Now ab is not parallel to s_i and intersects s_i at $a'' \neq a$. Thus, a and b are elements of P on opposite sides of the line l_i containing s_i.

21. A subset S of the plane is *star-shaped* from $a \in S$ if

$$\forall x \in S, \forall t \in [0,1], \quad ta + (1-t)x \in S,$$

that is, if for all points $x \in S$, the line segment ax lies in S.

(a) If S is convex, it is star-shaped from every $a \in S$. If S is star-shaped, it need not be convex. For example, $\{(x, y) \in [-1, 1] \times [-1, 1] \mid y = \pm x\}$ is star-shaped from $(0, 0)$, but not convex since, for example it contains $(.2, .2)$ and $(.2, -.2)$, but none of the points on the segment between these points.

10.2. Pick's Theorem

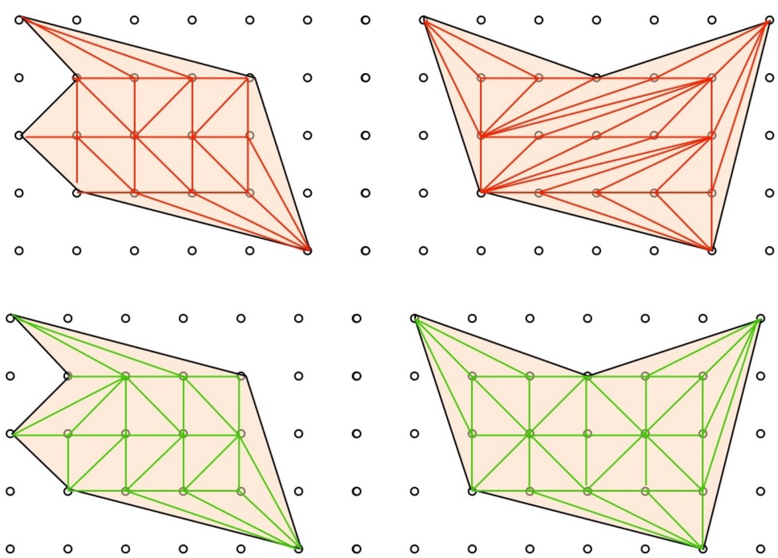
1. (a) The smallest value of s such that every square of side length s is guaranteed to contain a lattice point is $s = \sqrt{2}$. This is realized as a diamond having four lattice points $(m, n), (m+1, n), (m, n+1), (m+1, n+1)$ as midpoints of the sides. If s were any smaller, with the same center and orientation, the square would fall in the gaps between lattice points.

(b) The smallest value of s such that every square of side length s and sides parallel to the axes is guaranteed to contain a lattice point is $s = 1$. The square $[x, x+1] \times [y, y+1]$ must contain a lattice point for any $x, y \in \mathbb{R}$, since every interval $[x, x+1], [y, y+1]$ must contain an integer. If $s < 1$, the square $[(1-s)/2, 1-(1-s)/2]^2$ is a square of side length s which contains no lattice points. It falls in the open square $(0, 1)^2$.

6. For D, $(i, b) = (9, 6)$, so the area is 11.
For E, $(i, b) = (13, 5)$, so the area is 14.5.
For F, $(i, b) = (15, 6)$, so the area is 17.
For G, $(i, b) = (7, 11)$, so the area is 11.5.

13. A *fundamental triangle* is a lattice triangle which has no lattice points in its interior and the only lattice points on its boundary are the three vertices. Pick's theorem can be proved by showing that every simple lattice polygon can be dissected into fundamental triangles, and any way this is done will always use the same number of fundamental triangles. (A *dissection* of a simple polygon is defined just as a triangulation, dropping the requirement that the vertices of the triangles be vertices of the polygon and interpreting the triangle and polygon to be their enclosed areas.)

(a) For the lattice polygon D and E of Exercise 6, give two different dissections of each into fundamental triangles. How many fundamental triangles are used in these dissections?



Both dissections of D use 22 fundamental triangles. Both dissections of E use 29 fundamental triangles.

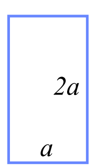
(b) What is the area of a fundamental triangle?
By Pick's theorem, the area of a fundamental triangle is $i + b/2 - 1 = 0 + 3/2 - 1 = 1/2$. Or, without using Pick's theorem, suppose we know every triangulation into fundamental triangles always uses the same number

of triangles. The smallest possible lattice triangles have area 1/2. Suppose T_1 is a fundamental triangle with area strictly greater than 1/2. Circumscribe it by a $n \times n$ square S with horizontal and vertical sides. Each of the n^2 1×1 squares in S can be triangulated into two fundamental triangles by adding a diagonal. Thus, any triangulation of S into fundamental triangles uses $2n^2$ triangles. By connecting the vertices of T_1 to the vertices of T_1, we can decompose S into simple lattice polygons which can be triangulated by fundamental triangles, so there exists a triangulation of S into $2n^2$ fundamental triangles $T_1, \ldots, T_{2n^2}$, including the triangle T_1. Since each T_i has area $\geq 1/2$ for $i = 2, \ldots, n$ and T_1 has area $> 1/2$, the sum of the areas of these fundamental triangles is strictly greater than $2n^2(1/2) = n^2$. This is a contradiction, since these triangles give a triangulation of the square S of area n^2. Thus, the area of a fundamental triangle T_1 must be 1/2.

(c) Given an arbitrary lattice polygon with i interior lattice points and b boundary lattice points, how many triangles are used in a dissection into fundamental triangles?
By Pick's theorem, the area of an arbitrary lattice polygon is $i + b/2 - 1$. We can dissect this into m fundamental triangles of area 1/2, so $m = 2i + b - 2$.

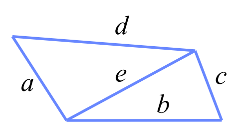
10.3. Irrational Approximations of π

7. Show that each polygon below can be repositioned using rigid motions (translations, reflections, rotations) so that all the vertices fall on a square lattice.



$a = \sqrt{13}$

(b)

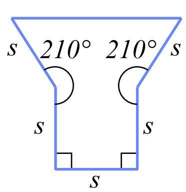


$a = \sqrt{13}, b = \sqrt{26}, c = \sqrt{10}, d = \sqrt{37}, e = 4\sqrt{2}$

(c)

(b) Vertices are $(0,3), (2,0), (8,4), (6,7)$, among others.
(c) Vertices are $(0,3), (2,0), (7,1), (6,4)$, among others.

8. Show that each polygon below cannot be repositioned using rigid motions (translations, reflections, rotations) so that all the vertices fall on a square lattice. Here, s denotes an integer.



The key observation here is that if we start at a vertex of a lattice polygon and translate it by the vector that has the same magnitude and direction as one of the sides of the polygon, then we again get a lattice point. Indeed, going from any lattice point to another involves adding $(j,k) \in \mathbb{Z}^2$ onto the coordinates of the first point. Equivalently, translating any side of a lattice polygon to any vertex of the lattice polygon gives another lattice point.

(b) Translate the bottom of the right angled side to the top left vertex and translate the bottom of the left angled side to the top right vertex. Now the resulting four angled sides together with two translations of the bottom side give a regular hexagon. Thus, if this figure is a lattice polygon, there is a regular hexagon which is a lattice polygon. This is impossible by Theorem 10.3.1.

9. Determine whether a triangle with sides of length a, b, c can be realized as a lattice polygon (on a 1×1 square lattice) for the values of a, b, c given. Justify your answers, and for those which can be realized as a lattice polygon, give the vertices of one realization.
   (a) One lattice triangle with side lengths $a = 5, b = \sqrt{109}, c = \sqrt{104}$ has vertices $(0,0), (5,0), (3,10)$
   (b) A lattice triangle with side lengths $a = 5, b = \sqrt{109}, c = \sqrt{103}$ is not possible. $\sqrt{103}$ is never realized as the distance between two lattice points since $103 \neq m^2 + n^2$ for any two integers m, n.

19. Suppose $\theta \in (0, \pi/2)$ is a rational multiple of π and $\theta \notin \{\pi/3, \pi/6\}$. By Theorem 10.3.2, $\cos\theta$ is irrational.
   (a) If $\theta \in (0, \pi/4) - \{\pi/6\}$ and is a rational multiple of π, then by Theorem 10.3.2, $\cos 2\theta$ is irrational, so $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ is irrational. Suppose $\theta \in (\pi/4, \pi/2) - \{\pi/3\}$. Then $2\theta \in (\pi/2, \pi) - \{2\pi/3\}$, and we may write $2\theta = \pi/2 + x$ where $x \in (0, \pi/2)$. By the symmetry of the cosine function, $\cos(\pi/2 + x) = -\cos(\pi/2 - x)$, and $\pi/2 - x \in (0, \pi/2)$, so $\cos(\pi/2 - x)$ is irrational unless $2\theta = 2\pi/3$. But, we have chosen $\theta \neq \pi/3$, so $\cos(2\theta)$ is irrational, and thus $\cos^2\theta$ is irrational.
   (b) $tan^2\theta = sec^2\theta - 1 = \frac{1}{\cos^2\theta} - 1$. By (a), $\cos^2\theta$ is irrational, so its reciprocal must be irrational, and then it follows that $tan^2\theta$ is irrational.If $\tan\theta$ were rational, then $\tan^2\theta$ would be rational. Thus, $\tan\theta$ is irrational.

10.5. Tiling and Visibility

5. If you want a regular tiling of the plane so that each cell contains 1 unit of area, and you want to minimize the perimeter of each cell, which tiling should you use? Where does this occur in nature?

   If A_n is the area of a regular n-gon with side s, then $A_3 = \sqrt{3}s^2/4$, $A_4 = s^2$, and $A_6 = 3\sqrt{3}s^2/2$. Now setting $A_n = 1$ and solving for the side length s_n of a regular n-gon with area 1, we find $s_3 = \sqrt{4/\sqrt{3}}$, $s_4 = 1$, $s_6 = \sqrt{2/(3\sqrt{3}}$. Now the perimeter p_n of the regular n-gon with area 1 is ns_n, so $p_6 = \sqrt{24}\sqrt{1/\sqrt{3}} < p_4 = \sqrt{16\sqrt{3}}\sqrt{1/\sqrt{3}} < p_3 = \sqrt{36}\sqrt{1/\sqrt{3}}$. Thus, the regular hexagon minimizes the perimeter among regular tilings with cells of 1 unit area. Bees, when constructing honeycomb, want to tile the plane with cells of fixed area using the least materials, so they use hexagonal tilings.

7. Randomly pick two unit squares from a square tiling of the plane, and let A be the union of these two squares. The plane can be tiled by translations of A.

   Case 1: The lower left corners of the squares in A are (a, b) and (m, b), where $a < m$. Now the horizontal translations $A+(t, 0)$ for $t = 0, 1, 2, \ldots, m-a-1$ tile a rectangle $[a, m+(m-a-1)] \times [0, 1]$. Now translations of this rectangle will tile the plane.

   Case 2: The lower left corners of the squares in A are (a, b) and (m, n), where $a < m$. Now the horizontal translations $A+(t, 0)$ for $t = 0, 1, 2, \ldots, m-a-1$ tile two rectangles $[a, m-1] \times [0, 1] \cup [m, m+(m-a-1)] \times [0, 1]$. Now vertical translations of these by $(0, j)$ $(j \in \mathbb{Z})$ tiles a vertical strip $[a, m+(m-a-1)] \times [\mathbb{R}$. Now horizontal translation of these by $(0, k(2m-2a))$ for $k \in \mathbb{Z}$ tiles the plane.

15. Given $n \in \mathbb{N}$, there are infinitely many horizontal rows $\{j+1, j+2, \ldots, j+n\} \times \{k\}$ of n adjacent lattice points, none of which are visible from the origin.

    Pick any natural number j and let $k = (j+1)(j+2)\cdots(j+n)$. Since $j+i$ is a divisor of k for $i = 1, \ldots n$, k and $j+i$ are not relatively prime for any $i = 1, \ldots, n$. Thus, no point in the rectangle $\{j+1, j+2, \ldots, j+n\} \times \{k\}$ is visible from the origin.

18. The proof of Theorem 10.5.6 gave $\zeta(2)$ as an infinite product. Similarly,

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \prod_{p\ prime} (1 + \frac{1}{p^3} + \frac{1}{p^6} + \frac{1}{p^9} \cdots).$$

$$\zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5} = \prod_{p\ prime} (1 + \frac{1}{p^5} + \frac{1}{p^{10}} + \frac{1}{p^{15}} \cdots).$$

10.6. Covering Properties and Geometry of Point Sets

3. The intersection of finitely many closed sets is closed. Suppose $C_1, \ldots, C_n$ are closed sets. To see that $C = C_1 \cap \cdots \cap C_n$ is closed, we will show that it contains all of its boundary points. Suppose x is a boundary point of C. Then for every $\epsilon > 0$, the ball $B(x, \epsilon)$ intersects $C = C_1 \cap \cdots \cap C_n$. Thus, $B(x, \epsilon)$ intersects C_i for every $\epsilon > 0$ and every $i = 1, \ldots, n$, so x is a boundary point of C_i for each $i = 1, \ldots, n$. Since each C_i is closed, each C_i contains all of its boundary points, so $x \in C_i$ for each $i = 1, \ldots, n$. This shows $x \in C = C_1 \cap \cdots \cap C_n$. Thus, C contains all of its boundary points, so C is closed.

7. For any five distinct points in the plane, there exist three which form an obtuse or straight angle. Consider the convex hull of the five points. If it is a pentagon, then the vertices of the pentagon are the five points. Since the interior angles of a convex pentagon measure less than π and their sum is 3π, the pigeonhole principle says one of the angles θ must satisfy $\pi > \theta \geq 3\pi/5 > \pi/2$. If any three of the (distinct) points are collinear, they form a straight angle. Suppose no three of the points are collinear and the convex hull of the five points is a quadrilateral or triangle. Then one of the points, say A, lies in the interior of a triangle formed by three of the vertices B, C, D of the convex hull. Now connecting A to B, C, and D forms three angles, each less than π,

whose sum is 2π, and again by the pigeonhole principle, at least one of the angles measures $\geq 2\pi/3$.

12. Dirichlet's approximation theorem also holds for rational numbers α, but getting rational approximations to a rational number is less interesting. If $\alpha = m/n$ is rational, then with $q_k = kn$ and $p_k = km$, we have $|\alpha - p_k/q_k| = 0 < 1/q_k^2$. This exhibits infinitely many rational approximations to $\alpha = m/n$ with small error—in fact, with zero error.

10.7. Linear Algebra and Packing the Plane

3. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ and D is a convex set in $\mathbb{R}^2$. Suppose $a, b \in T^{-1}(D)$ and $t \in [0,1]$. We wish to show $ta + (1-t)b \in T^{-1}(D)$, or equivalently, $T(ta + (1-t)b) \in D$. But $T(ta + (1-t)b) = tT(a) + (1-t)T(b)$, and since $T(a), T(b)$ are in the convex set D, we have $tT(a) + (1-t)T(b) = T(ta + (1-t)b) \in D$, which shows $T^{-1}(D)$ is convex.

5. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $S = [0,1] \times [0,1]$ is the unit square, and Q is any square with vertical and horizontal sides of length s. Show that $T(Q)$ is a translation of a copy of $T(S)$ scaled by s, and conclude that T scales the area of Q by the scale factor $d =$ the area of $T(S)$.

   Note that S has vertices $u_1 = (0,0)$, $u_2 = (1,0), u_3 = (1,1), u_4 = (0,1)$. Let Q be any arbitrary square with vertical and horizontal sides of length s, having vertices $v_1 = (x,y), v_2 = (x+s,y), v_3 = (x+s,y+s), v_4 = (x,y+s)$. We will show that for $i = 1,2,3,4,$, $T(v_i) = T(v_1) + sT(u_i)$.

   $T(v_1) = T(x,y) = (ax+cy, bx+dy) = (ax+cy, bx+dy) + s(0,0) = T(v_1) + sT(u_1)$
   $T(v_2) = T(x+s,y) = (ax+as+cy, bx+bs+dy) = (ax+cy, bx+dy) + s(a,b) = T(v_1) + sT(u_2)$
   $T(v_3) = T(x+s,y+s) = (ax+as+cy+cs, bx+bs+dy+ds) = (ax+cy, bx+dy) + s(a+c, b+d) = T(v_1) + sT(u_3)$
   $T(v_4) = T(x,y+s) = (ax+cy+cs, bx+dy+ds) = (ax+cy, bx+dy) + s(c,d) = T(v_1) + sT(u_v)$ This shows that $T(Q)$ is a translation (by $T(v_1)$) of a copy of $T(S)$ scaled by s (in each direction). It follows that the area of $T(Q)$ is s^2 times the area of $T(S)$, so T scales the area of every square Q with vertical and horizontal sides by the same amount, namely the area of $T(S)$.

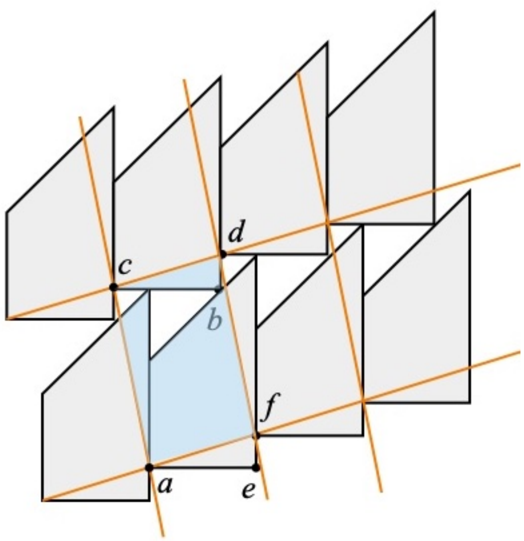
7. (a) The linear transformation T with $T((0,r)) = (0,a)$ and $T((r,0)) = (b,0)$ has matrix $M = \begin{pmatrix} a/r & 0 \\ 0 & b/r \end{pmatrix}$.
   (b) If C is a circle of radius $r > 0$ centered at the origin, it has area πr^2 and $\det M = ab/r^2$, so the area of $T(C)$ is $\pi r^2 |ab/r^2| = |a||b|r^2$.
   (c) $T(C)$ is an ellipse. The equation of the ellipse through $(0,a)$ and $(b,0)$ is $x^2/a^2 + y^2/b^2 = 1$. (Note that this holds whether a and b are positive or negative.) If a or $b = 0$, then $T(C)$ is not a (nondegenerate) ellipse, since $T(C)$ is not two-dimensional. Note that the following are equivalent: (i) $x^2 + y^2 = r^2$ $(r > 0)$, (ii) $x^2/r^2 + y^2/r^2 = 1$, and (iii) $(ax/r)^2/a^2 + (by/r)^2/b^2 = 1$. Thus, (x,y) lies on the circle $x^2 + y^2 = r^2$ if and only if

$T(x, y) = (ax/r, by/r)$ falls on the ellipse $x^2/a^2 + y^2/b^2 = 1$. Thus, $T(C)$ is an ellipse.

11. (a) Translate the vertices $(1, 4), (3, 7), (5, 2)$ by $(-1, -4)$ to $(0, 0), (2, 3), (4, -2)$ and apply Corollary 10.7.4 to get the area is $|-4 - 12|/2 = 8$.

16. Let T be the trapezoid having vertices $(0, 0), (1, 0), (1, 2)$, and $(0, 1)$. Find the packing density for a lattice translation packing of the plane by copies of T.

Let $a = (0, 0)$. The densest packing of such trapezoids will have a corner $b = (x, 1 + x)$ of a copy of T on the top edge of T, as seen in the figure. Note that $x \in [0, 1]$. Now c is one unit to the left of b, so $c = (x - 1, 1 + x)$. The point d is $(x, 2)$, so $bd = 2 - (1 + x) = 1 - x$. Since cbd is congruent to aef, we have $f = (1, 1 - x)$. Thus, the fundamental parallelogram determined by the edges af and ac is the image of $[0, 1]^2$ under the transformation with matrix $M = \begin{pmatrix} 1 & x - 1 \\ 1 - x & 1 + x \end{pmatrix}$ which has $T((1, 0)) = f = (1, 1 - x)$ and $T((0, 1)) = c = (x - 1, 1 + x)$. The area of the fundamental parallelogram is $|\det M| = x^2 - x + 2$, which is positive on $[0, 1]$. As seen in the figure, the fundamental parallelogram contains 3 parts of trapezoids which can be reassembled into one full copy of T. Since T has area $3/2$ and the area of the fundamental parallelogram is $x^2 - x + 2$, the trapezoids cover $(3/2)/(x^2 - x + 2)$ of the parallelogram, and since the parallelograms tile the plane, the lattice translation packing density of this packing is $(3/2)/(x^2 - x + 2)$. To maximize this density, we minimize the denominator. The parabola $y = x^2 - x + 2$ is minimum at its vertex $(1/2, 7/4)$. Thus, the maximum packing density is $(3/2)/(7/4) = 6/7 \approx 85.71\%$.



10.8. Helly's Theorem

5. Suppose $n \geq 3$, $C, A_1, \ldots, A_n$ are convex sets in the plane, and for any three sets A_i, A_j, A_k, there is a translation $x + C$ of C which intersects each of A_i, A_j, and A_k. Show that there is a translation of C which intersects each A_m, $m = 1, 2, \ldots, n$. (Compare to Exercise 3.)

Under the hypotheses, let $B_i = \{x \in \mathbb{R}^2 \mid x + C \cap A_i \neq \emptyset\}$. If each B_i is convex, the hypothesis that these sets are triple-wise overlapping implies, by Helly's theorem, that there is a point $x' \in B_1 \cap \cdots \cap B_n$, and then $x' + C$ intersects A_m, $m = 1, 2, \ldots, n$. It only remains to show each B_i is convex. Suppose $b_1, b_2 \in B_i$ and $t \in [0, 1]$. Then there exists $c_1, c_2 \in C$, $a_1, a_2 \in A_i$ such that $b_1 + c_1 = a_i$ and $b_2 + c_2 = a_2$. Multiplying these equations by t and $(1 - t)$ respectively and adding gives $tb_1 + (1 - t)b_2 + t(c_1) + (1 - t)c_2 = ta_1 + (1 - t)a_2$. By the convexity of C and A_i, $tb_1 + (1 - t)b_2 + c_3 = a_3$ for some $c_3 \in C, a_3 \in A_i$. Thus, $tb_1 + (1 - t)b_2 + C \cap A_i \neq \emptyset$, so $tb_1 + (1 - t)b_2 \in B_i$. This shows B_i is convex, as needed.

6. Suppose C is a convex set in the plane and C is covered by a finite collection of open half-planes $\{H_1, \ldots, H_n\}$. Then C is covered by some three of the half-planes. If $C \subseteq \bigcup_{i=1}^n H_i$, then $\bigcap_{i=1}^n C - H_i = C - \bigcup_{i=1}^n H_i = \emptyset$. Now $C - H_i = C \cap (\mathbb{R}^2 - H_i)$ is convex for each i. Since the intersection of all the sets $C - H_i$ is empty, there must be three of the sets $C - H_1, C - H_2, C - H_3$ with empty intersection, or else this contradicts Helly's theorem. Now if $\emptyset = \bigcap_{i=1}^3 C - H_i = C - \bigcap_{i=1}^3 H_i$, then $C - \subseteq \bigcap_{i=1}^3 H_i$, so $\{H_1, H_2, H_3\}$ covers C.

9. The vertices of the equilateral triangle in Figure 10.8.2 show that the bound $diam(F)/\sqrt{3}$ in Jung's theorem is sharp.

Continued Fractions

11.1. Finite Continued Fractions

2. (a) $[4;\,]$
   (c) $[4;10]$
   (f)

$$\frac{-23}{7} = -3\frac{2}{7} = -4+\frac{5}{7} = -4+\cfrac{1}{\left(\frac{7}{5}\right)} = -4+\cfrac{1}{1+\frac{2}{5}} = -4+\cfrac{1}{1+\cfrac{1}{\left(\frac{5}{2}\right)}}$$

$$= -4+\cfrac{1}{1+\cfrac{1}{2+\frac{1}{2}}} = [-4;1,2,2]$$

   (i)

$$\frac{-7}{5} = -2+\frac{1}{\left(\frac{5}{3}\right)} = -2+\cfrac{1}{1+\cfrac{1}{\left(\frac{3}{2}\right)}} = -2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}} = [-2;1,1,2]$$

3. (b) $[0;\,]=0$, $[0;2]=\frac{1}{2}$, $[0;2,5]=\frac{5}{11}$, $[0;2,5,4]=\frac{21}{46}$.
   (d) $[-4;\,]=-4$, $[-4;1]=-3$, $[-4;1,1]=\frac{-7}{2}$, $[-4;1,1,1]=\frac{-10}{3}$, $[-4;1,1,1,2]=\frac{-27}{8}$.

6. The expressions in (a) both equal $2\frac{3}{7}$; those in (b) both equal $2\frac{1}{4}$. The expressions on the left are not regular continued fractions (see the -2 in (a) and the second 2 in (b)), so Theorem 11.1.6 does not apply.

10.

k	$\frac{k}{14}$	$1-\frac{k}{14}=\frac{14-k}{14}$
1	[0;14]	[0;1,13]
2	[0;7]	[0;1,6]
3	[0;4,1,2]	[0;1,3,1,2]
4	[0;3,2]	[0;1,2,2]
5	[0;2,1,4]	[0;1,1,1,4]
6	[0;2,3]	[0;1,1,3]
7	[0;2]	[0;2] = [0;1,1]

For $0 < k < 7$, the continued fraction for $1-\frac{k}{14}$ is of form $[0; a_1, a_2, \ldots, a_n]$ where $n \geq 1$, and the continued fraction for $\frac{k}{n}$ is $[0; a_1 + a_2, \ldots, a_n]$. Furthermore, in this notation, $a_1 = 1$. For $n = 7$, the two representations $[0;2] = [0;1,1] = [0;1+1]$ for $\frac{7}{14}$ allow us to apply the pattern in this case as well. The pattern is described in general in Exercise 11.

11.2. Convergents of a Continued Fraction

1. No. a_0 is the integral part of $[a_0; a_1, \ldots, a_n]$. If $x \notin \mathbb{Z}$, the integral part of $-x$ is not the negative of the integral part of x. For example, $[2;3] = \frac{7}{3}$, so $-[2;3] = \frac{-7}{3}$. The integral part of $\frac{-7}{3}$ is -3, so the continued fraction for $-[2;3]$ is not of form $[-2; a_1, \ldots, a_n]$. In fact, $-[2;3] = [-3;1,2]$ and $[-2;3] = \frac{-5}{3}$.
2. (a) $C_0 = 0, C_1 = 1, C_2 = \frac{1}{2}, C_3 = \frac{3}{5}, C_4 = \frac{7}{12}, C_5 = \frac{10}{17}, C_6 = \frac{17}{29}$, $C_7 = \frac{44}{75}, C_8 = \frac{105}{179}$
   (b) $C_0 = 2, C_1 = 3, C_2 = \frac{11}{4}, C_3 = \frac{14}{5}, C_4 = \frac{67}{24}, C_5 = \frac{81}{29}, C_6 = \frac{472}{169}$, $C_7 = \frac{553}{198}$
3. Since $q_k = a_k q_{k-1} + q_{k-2}$, the q_k's will increase most slowly if every $a_k = 1$.
$$([1;1],[1;1,1],\ldots,[1;1,1,1,1,1,1]) = \left(\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13}\right).$$
   Each expression is of form $\frac{F_{n+2}}{F_{n+1}}$ where F_n is the n^{th} Fibonacci number.
7. (a) $\frac{225}{157} = \frac{p_4}{q_4} = [1;2,3,4,5]$; $[1;2,3,4] = \frac{43}{30} = \frac{p_3}{q_3}$; $[1;2,3] = \frac{10}{7} = \frac{p_2}{q_2}$; $[1;2] = \frac{3}{2} = \frac{p_1}{q_1}$; $[1;] = 1$.
   (b) $\frac{157}{30} = [5;4,3,2] = \frac{q_4}{q_3}$. $\frac{30}{7} = [4;3,2] = \frac{q_3}{q_2}$. $\frac{7}{2} = [3;2] = \frac{q_2}{q_1}$. $\frac{2}{1} = [2;] = \frac{q_1}{q_0}$.
   (c) Each is of form $\frac{q_k}{q_{k-1}}$.

11.3. Infinite Continued Fractions

3. $$1+\frac{1}{1+\sqrt{2}} = \frac{(1+\sqrt{2})+1}{1+\sqrt{2}} = \frac{2+\sqrt{2}}{1+\sqrt{2}}\left(\frac{1-\sqrt{2}}{1-\sqrt{2}}\right) = \frac{-\sqrt{2}}{-1} = \sqrt{2}.$$
   Putting this expression for $\sqrt{2}$ in place of the $\sqrt{2}$ appearing on the left gives
$$\sqrt{2} = 1+\cfrac{1}{1+1+\cfrac{1}{1+\sqrt{2}}} = 1+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}.$$
   Repeating this gives $\sqrt{2} = [1;2,2,2,2,\ldots] = [1;\overline{2}\,]$.

6. (a) If $r \neq 0$ is a root of $p(x) = ax^2 + bx + c$ $(a, b, c \in \mathbb{Z})$ and $k \in \mathbb{Z}$, then translating $p(x)$ by k units to the right gives a parabola with zero at $r + k$. That is, $r + k$ is a root of the polynomial $p(x - k)$. Furthermore, if $ar^2 + br + c = 0$ and $r \neq 0$, dividing by r^2 gives $a + b(\frac{1}{r}) + c(\frac{1}{r})^2 = 0$, so $\frac{1}{r}$ is a root of $q(x) = cx^2 + bx + a$.

   (b) Given a periodic continued fraction $[a_0; a_1, \ldots, a_{k-1}, \overline{a_k, \ldots, a_{k+j}}\,]$, Exercise 5 shows that $r_1 = [0; \overline{a_k, \ldots, a_{k+j}}\,]$ is a root of a quadratic equation with integer coefficients. Now $r_2 = [a_{k-1}; \overline{a_k, \ldots, a_{k+j}}\,] = a_{k-1} + \frac{1}{r_1}$ is also a root of a quadratic equation with integer coefficients by an application of part (a). Similarly,

$$r_3 = [a_{k-2}; a_{k-1}, \overline{a_k, \ldots, a_{k+j}}\,] = a_{k-2} + \frac{1}{r_2}$$

   is also a root of a quadratic equation with integer coefficients. Continuing this iterative process, we find that the original continued fraction $[a_0; a_1, \ldots, a_{k-1}, \overline{a_k, \ldots, a_{k+j}}\,] = r_{k+1}$ is a root of a quadratic equation with integer coefficients.

11.4. Applications of Continued Fractions

1. (a) Since 17 and 13 are relatively prime, Theorem 11.4.1 tells us that all solutions of the Diophantine equation $17x + 13y = 981$ have form $(21 + 13j, 48 - 17j)$ for $j \in \mathbb{Z}$. Four other solutions may be found by taking $j = 1, -1, 2$, and 10, giving solutions $(34, 31), (8, 65), (47, 14)$, and $(151, -122)$.

2. (a) $26x + 53y = 3938$: Since 26 and 53 are relatively prime, we find $\frac{26}{53} = [0; 2, 26]$, a continued fraction of order $n = 2$ with convergent $C_{n-1} = C_1 = \frac{1}{2}$. By Theorem 11.4.2, the solutions (x, y) have form

$$((-1)^1 3938(2) + 53k,\ (-1)^2 3938(1) - 26k) = (53k - 7876,\ 3938 - 26k).$$

   As we want positive solutions, $x = 53k - 7876 > 0$ implies $k > \frac{7876}{53} \approx 148.6$, and $y = 3938 - 26k > 0$ implies $k < \frac{3938}{26} \approx 151.5$. Thus, $k = 149, 150$, or 151, yielding solutions $(x, y) = (21, 64), (74, 38)$, and $(127, 12)$.

   (d) $213x + 121y = 6714$: Since 213 and 121 are relatively prime, we find $\frac{213}{121} = [1; 1, 3, 5, 1, 4]$, a continued fraction of order $n = 5$ with convergent $C_{n-1} = C_4 = \frac{44}{25}$. By Theorem 11.4.2, the solutions have form

$$\begin{aligned}(x, y) &= ((-1)^4 6714(25) + 121k,\ (-1)^5 6714(44) - 213k)\\ &= (167850 + 121k, -295416 - 213k).\end{aligned}$$

   As we want positive solutions, $x = 167850 + 121k > 0$ implies $k > \frac{-167850}{121} \approx -1387.2$, and $y = -295416 - 213k > 0$ implies $k < \frac{-295416}{213} \approx -1386.9$. Thus, $k = -1387$ and the only solution is $(x, y) = (23, 15)$.

5. Each congruence given has form $ax \equiv c \pmod{b}$ where a and b are relatively prime, so by Theorem 11.4.4, the solution is of form $x = (-1)^{n-1} c q_{n-1} + bk$ where $k \in \mathbb{Z}$, $\frac{a}{b} = [a_0; a_1, \ldots, a_n]$, and $[a_0; a_1, \ldots a_{n-1}] = \frac{p_{n-1}}{q_{n-1}}$.

(a) For $25x \equiv 18 \pmod{26}$, we find $\frac{25}{26} = [0; 1, 25]$, a continued fraction of order $n = 2$ and with $C_{n-1} = C_1 = 1$, so $q_1 = 1$ and all solutions are of form $(-1)^1 18(1) + 26k = -18 + 26k = 8 + 26k'$ $(k, k' \in \mathbb{Z})$. Thus, all solutions are congruent to 8 modulo 26.

7. Let x be the number of minutes the press ran. Then $28x$ newspapers were printed. Because enough newsprint for 65 newspapers remained on the last spool, the press had printed $235 - 65 = 170$ newspapers from its last spool. Thus, the number of newspapers printed is congruent to 170 modulo 235. That is, $28x \equiv 170 \pmod{235}$. Since $28 = 2^2 \cdot 7$ and $235 = 5 \cdot 47$ are relatively prime, Theorem 11.4.4 applies. We find that $\frac{28}{235} = [0; 8, 2, 1, 1, 5]$, a continued fraction of order $n = 5$. The first two convergents of this continued fraction are $C_0 = \frac{0}{1}$ and $C_1 = \frac{1}{8}$, so $q_0 = 1$ and $q_1 = 8$. From the recurrence relation $q_k = a_k q_{k-1} + q_{k-2}$, we find that $q_2 = 2(8) + 1 = 17$, $q_3 = 1(17) + 8 = 25$, and $q_4 = q_{n-1} = 1(25) + 17 = 42$. Since the solutions are congruent modulo 235 to $(-1)^{n-1} c q_{n-1} = (-1)^4 (170)(42) = 7140 = 90 + 235(30)$ and only positive answers are possible, the solution set is $\{90 + 235j | j \in \mathbb{N}\}$. The operator should not have spent $90 + 235$ minutes at lunch, so the only possible answer in the appropriate range is 90 minutes.

9. We find that $C_4 = \frac{134}{35}$ and $C_5 = \frac{1229}{321}$. Since all even convergents of α are below α, C_4 is an underestimate and $|\alpha - C_4| = \alpha - C_4$. Now Lemma 11.4.6 gives

$$\frac{1}{12,460} = \frac{1}{35(35+321)} < \alpha - \frac{134}{35} < \frac{1}{35(321)} = \frac{1}{11,235} \approx 0.000089.$$

It follows that

$$\frac{47,705}{12,460} = \frac{1}{35(35+321)} + \frac{134}{35} < \alpha < \frac{134}{35} + \frac{1}{35(321)} = \frac{1229}{321},$$

so $\alpha \in (\frac{47,705}{12,460}, \frac{1229}{321}) \subset (\frac{134}{35}, \frac{1229}{321}) = (C_4, C_5)$.

11. (b) $3\pi = [9; 2, 2, 1, 4, 1, 1, 1, 97, 4, \ldots]$ has convergent $C_5 = \frac{377}{40}$, and this must be the best approximation to 3π by a rational number with denominator ≤ 40.