

1 The basic equations of fluid dynamics

The main task in fluid dynamics is to find the velocity field describing the flow in a given domain. To do this, one uses the basic equations of fluid flow, which we derive in this section. These encode the familiar laws of mechanics:

- conservation of mass (the continuity equation, Sec. 1.2)
- conservation of momentum (the Cauchy equation, Sec. 1.3)

at the level of “fluid elements”, defined in Sec. 1.1. In any domain, the flow equations must be solved subject to a set of conditions that act at the domain boundary, Sec. 1.5. If the flow leads to compression of the fluid, we must also consider thermodynamics:

- conservation of energy.

However we defer this complication until later in the course, Sec. 5, assuming initially that the flow remains incompressible, Sec. 1.4.

1.1 The continuum hypothesis; fluid elements

At a microscopic scale, fluid comprises individual molecules and its physical properties (density, velocity, *etc.*) are violently non-uniform. However, the phenomena studied in fluid dynamics are macroscopic, so we do not usually take this molecular detail into account. Instead, we treat the fluid as a continuum by viewing it at a coarse enough scale that any “small” fluid element actually still contains very many molecules. One can then assign a local bulk flow velocity $\mathbf{v}(\mathbf{x}, t)$ to the element at point \mathbf{x} , by averaging over the much faster, violently fluctuating Brownian molecular velocities. Similarly one defines a locally averaged density $\rho(\mathbf{x}, t)$, *etc.* These locally averaged quantities then vary smoothly with \mathbf{x} on the macroscopic scale of the flow.

1.2 Conservation of mass

1.2.1 The continuity equation

Consider a volume V bounded by a surface S that is fixed in space. This mass inside it is given by $\int_V \rho dV$, so the

$$\text{rate of decrease of mass in } V = -\frac{d}{dt} \int_V \rho dV = -\int_V \frac{\partial \rho}{\partial t} dV. \quad (1)$$

If mass is conserved, Eqn. 1 must equal the total rate of mass flux out of V . How do we calculate this? The rate of outward mass flux across any small element $d\mathbf{S}$ of S is $\rho \mathbf{v} \cdot d\mathbf{S}$ where the magnitude of $d\mathbf{S}$ is equal to the element’s area and we take $d\mathbf{S}$ along the outward normal. Integrating over the whole surface we have

$$\text{rate of mass flux out of } V = \int_S \rho \mathbf{v} \cdot d\mathbf{S} = \int_V \nabla \cdot (\rho \mathbf{v}) dV \quad (2)$$

where we used Green’s formula to convert to a volume integral. The integrand $\nabla \cdot (\rho \mathbf{v})$ on the RHS is expressed in Cartesian coordinates $\mathbf{x} = (x, y, z)$, $\mathbf{v} = (u, v, w)$ as

$$\nabla \cdot (\rho \mathbf{v}) = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}. \quad (3)$$

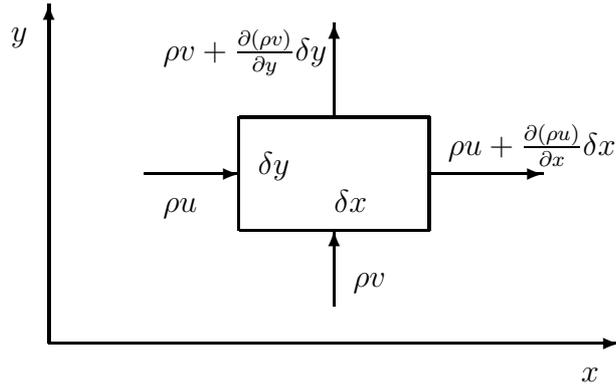


Figure 1: Mass fluxes entering and leaving an element.

See Fig. 1, which shows clearly that gradients in the flow field are required for non-zero net flux. For mass to be conserved everywhere, Eqns. 1 and 2 must be equal for any volume V and so we get

$$\text{the continuity equation: } \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (4)$$

1.2.2 The material derivative

The continuity equation contains the “time-derivative” of the fluid density. What does this mean exactly? For any physical quantity $f = f(\mathbf{x}, t)$ (density, temperature, each velocity component, *etc.*), we must actually take care to distinguish two different time derivatives. By $\partial f / \partial t$, as in Eqn. 4, we mean the rate of change of f *at a particular point that is fixed in space*. But we might instead ask about the rate of change of f *in a given element of fluid* as it moves along its trajectory $\mathbf{x} = \mathbf{x}(t)$ in the flow. This defines the material (or substantive) derivative

$$\begin{aligned} \frac{Df}{Dt} &= \frac{d}{dt} f(x(t), y(t), z(t), t) \\ &= \frac{\partial f}{\partial t} + \frac{dx}{dt} \frac{\partial f}{\partial x} + \frac{dy}{dt} \frac{\partial f}{\partial y} + \frac{dz}{dt} \frac{\partial f}{\partial z} \\ &= \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} \\ &= \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \end{aligned} \quad (5)$$

This. 5 conveys the intuitively obvious fact that, even in a time-independent flow field ($\partial f / \partial t = 0$ everywhere), any given element can suffer changes in f (via $\mathbf{v} \cdot \nabla f$) as it moves from place to place.

Check as an exercise that the continuity equation can also be written in the form

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \quad (6)$$

1.2.3 Incompressible continuity equation

If the fluid is **incompressible**, $\rho = \text{constant}$, independent of space and time, so that $D\rho/Dt = 0$. The continuity equation then reduces to

$$\nabla \cdot \mathbf{v} = 0, \quad (7)$$

which in Cartesian coordinates is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (8)$$

1.3 Conservation of momentum

1.3.1 The Cauchy equations

Consider a volume V bounded by a *material* surface S that moves with the flow, always containing the same material elements. Its momentum is $\int_V dV \rho \mathbf{v}$, so:

$$\text{rate of change of momentum} = \frac{d}{dt} \int_V dV \rho \mathbf{v} = \int_V dV \rho \frac{D\mathbf{v}}{Dt}. \quad (9)$$

(The mass ρdV of each material element is constant.) This must equal the net force on the element. Actually there are two different types of forces that act in any fluid:

- Long ranged external **body** forces that penetrate matter and act equally on all the material in any element dV . The only one considered here is gravity, $\rho \mathbf{g} dV$.
- Short ranged molecular forces, internal to the fluid. For any element, the net effect of these due to interactions with other elements acts in a thin **surface** layer. In 3D, each of the 3 sets of surface planes bounding an element experiences a 3-component force, giving 9 components in all. These form the stress tensor $[\Pi]$, defined so the force exerted per unit area across a surface element $d\mathbf{S} \equiv \hat{\mathbf{n}} dS$ (by the fluid on the side to which $\hat{\mathbf{n}}$ points *on* the fluid on the other side) is $\mathbf{f} = [\Pi] \cdot \hat{\mathbf{n}}$.

$$\begin{aligned} \text{Total force (body + surface)} &= \int_V dV \rho \mathbf{g} + \int_S [\Pi] \cdot d\mathbf{S} \\ &= \int_V dV (\rho \mathbf{g} + \nabla \cdot [\Pi]). \end{aligned} \quad (10)$$

By Newton's second law, Eqns. 9 and 10 must be equal for any V , so we get finally

$$\text{the Cauchy equation: } \rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} + \nabla \cdot [\Pi]. \quad (11)$$

The physical meaning of this is seen clearly in Cartesian coordinates (in 2D in Fig. 2):

$$\begin{aligned} \text{Momentum, } x : \rho \frac{Du}{Dt} &= \rho g_x + \frac{\partial}{\partial x}(\Pi_{xx}) + \frac{\partial}{\partial y}(\Pi_{xy}) + \frac{\partial}{\partial z}(\Pi_{xz}) \\ \text{Momentum, } y : \rho \frac{Dv}{Dt} &= \rho g_y + \frac{\partial}{\partial x}(\Pi_{yx}) + \frac{\partial}{\partial y}(\Pi_{yy}) + \frac{\partial}{\partial z}(\Pi_{yz}) \\ \text{Momentum, } z : \rho \frac{Dw}{Dt} &= \rho g_z + \frac{\partial}{\partial x}(\Pi_{zx}) + \frac{\partial}{\partial y}(\Pi_{zy}) + \frac{\partial}{\partial z}(\Pi_{zz}). \end{aligned} \quad (12)$$

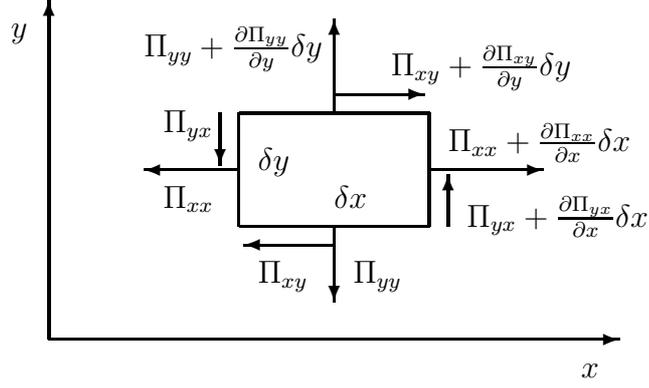


Figure 2: Surface stresses on a fluid element in 2 dimensions.

Π_{ij} is the force per unit area in the i direction across a plane with normal in the j direction. As can be seen from the figure, *gradients* in the stress tensor are needed for there to be a net force on any element (consistent with the surface integral of $[\Pi] \cdot \hat{\mathbf{n}}$ equating to a volume integral of $\nabla \cdot [\Pi]$). It is possible to show that the stress tensor is symmetric, *i.e.*

$$\Pi_{xy} = \Pi_{yx}, \quad \Pi_{zx} = \Pi_{xz}, \quad \Pi_{yz} = \Pi_{zy}, \quad (13)$$

otherwise any small fluid element would suffer infinite angular acceleration.

1.3.2 Constitutive relations

The surface stresses $[\Pi]$ on any element arise from a combination of pressure p and viscous friction, as prescribed by the constitutive relations

$$\Pi_{xx} = -p + \lambda \nabla \cdot \mathbf{v} + 2\mu \frac{\partial u}{\partial x}, \quad \Pi_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad (14)$$

$$\Pi_{yy} = -p + \lambda \nabla \cdot \mathbf{v} + 2\mu \frac{\partial v}{\partial y}, \quad \Pi_{yz} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \quad (15)$$

$$\Pi_{zz} = -p + \lambda \nabla \cdot \mathbf{v} + 2\mu \frac{\partial w}{\partial z}, \quad \Pi_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right). \quad (16)$$

μ and λ are the coefficients of dynamic and bulk viscosity respectively. These expressions assume that the relationship between stress and velocity gradients is

- linear (which is valid for Newtonian fluids) and
- isotropic (*i.e.*, the intrinsic properties of the fluid have no preferred direction).

1.3.3 Incompressible Navier-Stokes equations

For incompressible flow, $\nabla \cdot \mathbf{v} = 0$ (Eqn. 7). The constitutive relations then reduce to

$$\Pi_{ij} = -p \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (17)$$

Here we have used suffix notation $\mathbf{v} = (u_1, u_2, u_3)$, $\mathbf{x} = (x_1, x_2, x_3)$ and defined the Kronecker delta symbol $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Check that you are happy

with this notation by working through the derivation of Eqn. 17 from Eqns. 14 to 16. Inserting Eqn. 17 into Eqn. 11, assuming constant μ , and utilising again the incompressibility condition 7, we get the incompressible Navier–Stokes (N–S) equations:

$$\begin{aligned}
\text{Continuity : } 0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
\text{Momentum, } x : \rho \frac{Du}{Dt} &= \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\text{Momentum, } y : \rho \frac{Dv}{Dt} &= \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\text{Momentum, } z : \rho \frac{Dw}{Dt} &= \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{aligned} \tag{18}$$

or, in compact notation

Continuity

$$\nabla \cdot \mathbf{v} = 0 \tag{19}$$

Momentum

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \Delta \mathbf{v}, \tag{20}$$

in which Δ is the Laplacian operator. For uniform ρ we can simplify Eqn. 20 by realising that the gravitational force is exactly balanced by a pressure gradient $\nabla p_0 = \mathbf{g}$ that does not interact with any flow: defining $P = p - p_0$, we get

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \mu \Delta \mathbf{v}. \tag{21}$$

1.3.4 Physical interpretation

The surface stresses (pressure and viscous effects) on any fluid element were introduced above via the constitutive relation. What is their physical interpretation?

- Viscous stresses are generated by velocity gradients. They oppose relative motion of fluid elements. Consider Fig. 3. Across any plane AB, a tangential viscous stress $\Pi_{xy} = \mu \frac{\partial u}{\partial y}$ acts: the faster fluid above AB drags the fluid below forward, and the slower fluid below drags the fluid above back.
- Even in the absence of velocity gradients, each element still experiences an isotropic pressure p . In rest at equilibrium, this equals the thermodynamic pressure in the equation of state. In flow this is no longer true: p is now defined in purely mechanical terms, as a measure of the local intensity of ‘squeezing’ in the fluid.

1.4 Condition for incompressibility

We have seen that the continuity equation, Eqn. 4, and the constitutive relations, Eqns. 14 to 16, take on much simpler forms (Eqns. 7, 17) when the flow is incompressible. We therefore assume incompressibility for much of the course (deferring compressible flow to Sec. 5). The criterion¹ for this is that the flow speed U should be much less than the speed of sound a . In practice, this only breaks down for high speed (subsonic and hypersonic) gas flows.

¹Specifically, the Mach number $\text{Ma} \equiv U/a$ must obey $\text{Ma}^2 \ll 1$. (Tritton 5.8 for those interested.)

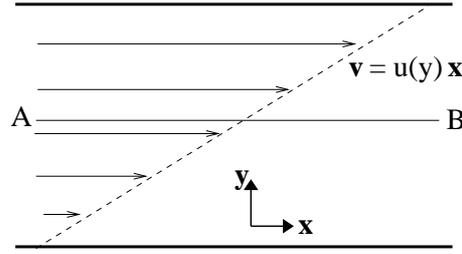


Figure 3: Planar shear flow.

1.5 Boundary conditions

In any flow domain, the flow equations must be solved subject to a set of conditions that act at the domain boundary. For a rigid bounding wall moving at velocity \mathbf{U} and having unit normal $\hat{\mathbf{n}}$, we assume for the local fluid velocity \mathbf{v} that

1. The wall is impermeable: $\mathbf{v} \cdot \hat{\mathbf{n}} = \mathbf{U} \cdot \hat{\mathbf{n}}$.
2. The fluid doesn't slip relative to the wall: $\mathbf{v} \times \hat{\mathbf{n}} = \mathbf{U} \times \hat{\mathbf{n}}$.

Condition 2 is not obvious: why shouldn't slip occur? The underlying notion is that the fluid interacts with the wall in the same way as with other fluid: there cannot exist any discontinuity in velocity, or an infinite viscous stress would arise. But the ultimate justification comes from experimental verification.

1.6 Summary

In this section, we have derived the basic equations governing incompressible fluid flow. In what follows, we are mainly concerned with flow fields that are time-independent and two-dimensional. Eqns. 19 and 21 then reduce, in Cartesian coordinates (x, y) to:

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (22)$$

Momentum:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (23)$$

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (24)$$

where $\mathbf{v} = (u, v)$ is the velocity field. As discussed above, the first term on the RHS in Eqns. 23 and 24 refers to *pressure* forces, $\nabla \cdot P$. The rest of the RHS describes *viscous* forces, $\mu \nabla^2 \mathbf{v}$. The LHS is the momentum change that any element experiences as it moves between regions of different velocity in the flow field. This has the dimensions of a force, and is referred to as the *inertia* force, $\rho \mathbf{v} \cdot \nabla \mathbf{v}$.

We will refer back to these basic equations 22 to 24 extensively throughout the rest of the course.